Investigating Student Understanding for a Statistical Analysis of Two Thermally Interacting Solids

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Abstract. As part of an ongoing research and curriculum development project for upper-division courses in thermal physics, we have developed a sequence of tutorials in which students apply statistical methods to examine the behavior of two interacting Einstein solids. In the sequence, students begin with simple results from probability and develop a means for counting the states in a single Einstein solid. The students then consider the thermal interaction of two solids, and observe that the classical equilibrium state corresponds to the most probable distribution of energy between the two solids. As part of the development of the tutorial sequence, we have developed several assessment questions to probe student understanding of various aspects of this system. In this paper, we describe the strengths and weaknesses of student reasoning, both qualitative and quantitative, to assess the readiness of students for one tutorial in the sequence.

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INTRODUCTION

Thermodynamics and statistical physics are core subjects of upper-division physics. A relatively new and increasingly popular course format is described as ‘thermal physics,’ which intermingles classical thermodynamics and statistical approaches. In this study, we examine some of the underlying reasoning that is necessary for one such topic, and probe student understanding of the situation.

BACKGROUND FOR THE RESEARCH

The second law of thermodynamics is well known as a challenging and complex idea, and PER on the law supports this notion [1, 2, 3]. An innovative instructional approach, found in several recent textbooks, motivates the second law of thermodynamics through an extended logical sequence that involves considering statistical phenomena.

Students first consider probability and determine a means of counting states for the Einstein model of a solid. This model is then applied to a system of two interacting solids of the same material but different mass. Students are shown that the expected macroscopic outcome (energy shared by the solids in proportion to their respective masses) is the energy arrangement of maximum probability. They then consider larger and larger systems, and apply statistical techniques to show that it is increasingly probable that the system is near the classical equilibrium state. As the number of particles approaches Avogadro’s number, the probability of the system being far from that equilibrium state becomes vanishingly small. Thus, if solids of different temperatures are placed together and allowed to exchange energy, they tend to evolve toward a maximum probability state, in which the average energy per particle for each solid is equal.

There has been relatively little published work on student understanding of statistical physics. The author and collaborators in this project have previously described studies focusing on aspects of this sequence [4, 5]. In addition, there are a number of relevant previous studies of student learning of probability in other contexts [6, 7, 8, 9].

This research seeks to probe student understanding of the reasoning tasks underlying this approach. It proceeds from the assumption that students construct understanding of scientific phenomena, in some cases developing ideas that are in contrast with accepted scientific viewpoints. We have sought to document student understanding of the target ideas using standard methods of PER, particularly written conceptual questions. The guiding research question has been:

To what extent do students in the thermal physics course utilize appropriate reasoning elements to analyze the statistical system of an Einstein solid and two interacting solids?

Context for Research

This project was performed in several sections of an upper-division thermal physics course at California State University Fullerton (CSUF), a large public comprehensive university serving a diverse student population. The course follows the hybrid 'thermal physics' approach described above, using a popular text [10] that develops the ideas of entropy and the second law of thermodynamics through a statistical approach. The course meets for two 75-minute blocks per week. Enrollments have ranged between 6 and 19, and typically a significant portion of class time is spent on small-group tutorial exercises.

Most students in the course are physics majors or minors who have completed introductory physics and several semesters of calculus. The CSUF introductory physics sequence does not include thermodynamics, but many students reported studying thermal physics in high school, in physics courses at other institutions, or in chemistry. A few students (10-20%) had previously completed a college-level math course in probability and statistics.

INITIAL STUDENT UNDERSTANDING

As a probe of initial student understanding, the author posed questions on an ungraded quiz. This quiz was given to students in class (three sections, $N = 38$) after completion of a tutorial, *Counting States*, which focuses on the mathematics of flipping coins and the binomial formula. After the quiz, students then worked through the second in a sequence of tutorials, described further below.

The questions on the ungraded quiz covered three topics: microstates for a system analogous to the Einstein solid (balls in boxes), the relationship of the multiplicity of a combined system to the multiplicities of the component parts, and comparison of probabilities for two ball/box situations.

Boxes and balls

The first problem focuses on the determination of multiplicity (i.e, the counting of microstates) for a system of balls in distinguishable boxes, with the statement that each box can hold any number of balls. This situation is novel for students, but is statistically similar to the Einstein solid, without the physics baggage of that system. Just as the boxes can contain multiple balls, each oscillator in an Einstein solid can have any number of energy units.

In the first part of this problem, students are asked how many ways there are to place one ball in four boxes. As the ball could go in box 1, box 2, box 3, or box 4, and the boxes are distinguishable, there are four possibilities. This part of the question has been answered correctly by almost 100% of students, suggesting that students are developing appropriate reasoning for counting microstates.

The second part of the question is more difficult, and asks how many ways there are to place two balls in the same four boxes, with the note that the balls are assumed to be indistinguishable. (In other words, if there is one ball in box 1 and one ball in box 3, that is a single state, as the identities of the two balls cannot be determined; if the balls were instead distinguishable, say with labels A and B, there would be two states, one with ball A in box 1 and ball B in box 3, and other with the locations reversed.) The second part, as noted, is more difficult. The correct answer is ten; there are four arrangements with two balls in a box (one for each box) plus six arrangements for placing one ball in two of the four boxes. (The latter arrangements could be described as 12, 13, 14, 23, 24, and 34, where the notation 'xy' indicates a ball in box x and a ball in box y.) A formula for this outcome can be used as well; this formula is derived in the course text as well as the tutorial sequence and will be described briefly below.

Only approximately half of the students answered the second part correctly. Students who answered correctly wrote a variety of supporting ideas on their paper. Many drew diagrams to enumerate all possible states. Others gave short written explanations similar to the one given above, but many gave no explanation. Interestingly, although there is a formula that students could use, no students have explicitly written out this formula as part of their response.

The incorrect answers on this problem included one common and eminently reasonable wrong answer. About a third of the students answered 6; this answer is 'four choose two,' the number of arrangements in which two boxes have one ball and two have none. Such an answer is perhaps not surprising given that students had recently completed the *Counting States* tutorial that develops the binomial formula. This answer is doubly compelling for students who enumerate states, but neglect the possibility that two balls could be in a single box. Some of these answers, though incorrect, reflect thoughtful and sophisticated thinking in which students arrive at a result and check for consistency, with one method involving a formula (though incorrect) and one involving a more diagrammatic response. We believe that these answers reflect students who are able to use many of the appropriate resources for this situation but are missing one or more key pieces. They may also indicate that the *Counting States* tutorial or other class instruction has helped students to develop some of the resources they will need to master this material.

Combining multiplicities

The second problem on the ungraded quiz seems simple; students are asked to express the multiplicity of a combined system of two Einstein solids in terms of the multiplicities of the two parts. The correct answer is that the multiplicity of the combined system is the product of the individual multiplicities: $\Omega_{combined} = \Omega_A \, \Omega_B$. The arrangement of energy in one solid cannot influence the arrangement in the other; for any arrangement in solid A, there are Ω_B arrangements for solid B. (This result combined with the statistical definition of entropy $S = k_B \ln \Omega$, indicates that the total entropy is the sum of the entropies of the constituents, so that entropy is an extensive variable.)

This question might appear trivial, but this relationship is a crucial link in the development of the model of interacting solids. If multiplicities were additive rather than multiplicative, the behavior of the system would be quite different. Student responses suggest that this step in the development should not be taken for granted; about 25% of the students answered correctly. The dominant wrong answer (about 60% of students) was to add multiplicities, with a few suggestions to take the arithmetic average of the two.

Combined balls and boxes

The final part of the initial ungraded quiz involved comparing the probability of two arrangements of balls in distinguishable boxes. Three blue boxes and three red boxes are arranged so they are distinguishable, and two balls are added randomly; as before, each box could hold any number of balls. Students are given descriptions of two macrostates: in State B, one ball is in a red box and the other is in a blue box, and in State A, both are in red boxes. Students are asked to compare the probabilities of these states. This problem is analogous to the problem of two interacting Einstein solids, with, for example, the three red boxes corresponding to a single 3-oscillator solid.

The correct answer for this part requires combining the reasoning for the previous two parts; for each state students can determine the number of arrangements for the blue boxes and the number for the red, and then multiply the two. For State B, there are three ways to place one ball in one of three red boxes, and three ways to place one ball in one of three blue boxes, so the combined system has nine arrangements. For State A, there is only one arrangement for the blue boxes, but six for the red (11, 22, 33, 12, 13, 23), and the combined system has six possibilities. Therefore State B, with the balls split between red and blue, is more probable, just as the most probable state for interacting Einstein solids is the one in which energy per particle is equal, and the solids are in thermal equilibrium.

As might be expected based on responses to the previous parts of the ungraded quiz, this problem proved to be quite difficult for students. About 15% of the students gave correct responses, many of which were accompanied by diagrams and/or multiplicities (which were not explicitly asked for in the problem statement). The most common incorrect answer (40% of students) was to state that the probabilities were equal. Though some of these answers were not accompanied by explanations, those that did include explanations typically referred to the assumption that all accessible microstates of a system are equally probable. These answers may reflect continuing difficulty for students in distinguishing macrostates from microstates, as described in the previous paper on this project [4].

As in the previous part, some of the incorrect answers nevertheless indicated that students bring appropriate ideas to bear on these problems. Several wrong answers included diagrams that attempted to enumerate all the states. Once again, some students neglected to consider states in which two balls were in the same box, thus finding equal multiplicities.

TUTORIAL SEQUENCE

The tutorial *States in the Einstein Solid* [11] was developed before the pretest and has been used several times, with minor variations, in the thermal physics course. The tutorial falls in approximately the seventh week of the course, after students have been introduced to the model for an Einstein solid in lecture and the textbook. The tutorial seeks to build upon and refine the ideas developed in the *Counting States* tutorial. It is therefore promising to note that the pretest results suggest that students are using ideas from the previous tutorial and that even incorrect answers reflect the influence of previous instruction.

The tutorial begins by reviewing the language and notation associated with the Einstein solid, which includes *N* independent oscillators and a number of energy units *q*, each corresponding to a single excitation of one harmonic oscillator. (The results assume a quantized oscillator, which is not explicitly covered in previous courses, but most students have been introduced to modern physics and quantization in introductory physics and chemistry, and this aspect of the model does not appear to be problematic for students.) The tutorial then guides students to recognize that the binomial formula from the *Counting States* tutorial cannot be applied in a simple way to the Einstein solid; the multiplicity cannot be '*N* choose *q*' because the system is not a binary system, as each oscillator can have any number of energy units.

Students then are guided to describe a microstate by drawing a diagram with vertical lines as boundaries between oscillators and dots as energy units [10]. For example, consider a solid with four oscillators (or the four boxes on the pretest), and two energy units (or two balls). Four oscillators require three boundaries, so diagrams will have three lines and two dots. A macrostate with two energy units in the first oscillator and none in the other three could be shown as ••|||. A state in which the first and last had one energy unit each would be represented as •|||•.

The students are then guided to see that each arrangement has the same number of characters, each of which is a dot or a line. They are asked to explicitly map this to the coin flipping system, indicating what quantity is analogous to the number of coins (number of symbols), and what quantities are analogous to numbers of heads or tails (number of energy units, q, or number of boundaries, $N - 1$). The students can then readily reproduce the multiplicity formula given in the textbook, $\Omega = \frac{(q+N-1)!}{q!(N-1)!}$. The final section of the tutorial leads students to the relationship for multiplicity for two interacting solids. The students calculate multiplicities for two small solids in contact and compare the probabilities of two macrostates, one with all energy in one solid, one with a more even distribution. Students find that the latter is more probable; this idea will return in subsequent tutorials in the instructional sequence.

ASSESSMENT AND CONCLUSION

The primary purpose of the tutorial is to help students develop tools to allow them to determine multiplicities and probabilities for a single Einstein solid and for a system of two interacting Einstein solids. The pretest results do indicate that many aspects of the tutorial are well-matched to student needs. For example, some students inappropriately apply the result for coin flipping directly to the Einstein solid, suggesting the need for an exercise in which students are led to recognize the difference in these systems. The portion of the tutorial that is intended to establish that multiplicity is multiplicative for the combined system has been a sticking point for students and typically requires instructor intervention, which is not surprising in light of the difficulty of this question on the pretest.

In order to assess the initial version of the tutorial, we have posed problems on course examinations, including a mix of qualitative and quantitative problems. For the Einstein solid, students have typically been asked to determine quantitative results, either multiplicity, probability, or entropy, and there is almost always a problem of this nature on the second course midterm. For example, students were asked to determine the most probable energy distribution for 12 energy units shared between Einstein solids with 12 and 6 oscillators and to determine the multiplicity.

After completion of the tutorial, 90% of the students ($N = 18$) correctly multiplied the multiplicities of the two solids, a significant improvement over the pretest results, and a similar fraction of students correctly applied the multiplicity formula. However, only about 30% of the students gave a completely correct answer. The most common type of error, including just over 40% of students, was based on incorrect splits of the energy units; the most probable macrostate will have a 2:1 ratio of energy units, or 8 units in the larger solid and 4 in the smaller. Most students making this error split the energy equally (appropriate for the equal size solids from the tutorial, but not for this case), and others miscalculated the 2:1 ratio. These errors clearly suggest that additional attention is needed in making the connection between the statistical model and macroscopic notion of thermal equilibrium for the interacting solids.

While the numbers of students are still small, classroom observation and pre- and posttest results suggest that the tutorial is only a mixed success. Based on these results, the tutorial will be revised and assessed further in the coming year.

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