Students' Understanding of the Concepts of Vector Components and Vector Products

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Abstract. In this article we investigate students' understanding of: 1) vector components and, 2) vector products. We administered a test to 409 students completing introductory physics courses at a private Mexican university. In the first part, based on the work of Van Deventer [1], we analyze the understanding of components of a vector. We used multiple-choice questions asking for students' reasoning to elaborate on the misconceptions and difficulties of graphical representation of the *x*- and *y*-components of a vector. In the rest of this work, we analyze the understanding of the dot and cross products. We designed opened-ended questions to investigate the difficulties on the calculation and the misconceptions in the interpretation of these two products.

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INTRODUCTION

In recent years, researchers have studied students' understanding of the vector component [1-3], the dot product [1-4], and the cross product [1, 3, 4] concepts. In this article, we present results of a study that contributes on further understanding of the difficulties and misconceptions on these three concepts.

In the first part, based on the work of Van Deventer [1], we investigate the understanding of the component concept. We study the misconceptions and difficulties with graphical representation of components by presenting and discussing results of students' answers and some excerpts of their reasoning.

In the second and third parts, we designed opened-ended questions to analyze the difficulties to calculate vector products and questions to investigate the misconceptions in the interpretation of these products. Knight [3] used similar calculation questions; however, he administered them to students entering university and found that less than 10% of students could solve these types of questions. In contrast, we administered the calculation and interpretation questions to students having completed all introductory physics courses.

This article covers three objectives: to analyze 1) the misconceptions about vector component concepts, 2) the misconceptions and difficulties with the dot product, and 3) the misconceptions and difficulties with the cross product. In the following section, we present the details of the methodology of this study. Later, we divided the Results and Discussion section in subsections covering each of the three objectives. At the end, we present our conclusions and a review of the main results of the study.

METHODOLOGY

This research was conducted in a large private Mexican university. Multiple-choice questions with reasoning explanation and open-ended questions were administered to 409 students in their last of three calculus-based introductory physics courses at this institution.

Fig. 1 shows the questions used in this study. To address the first objective, we used Questions 1 and 2 which are modifications of those designed by Van Deventer [1]. (More details below.) We designed Questions 3 and 5 to address the second objective, and Questions 4 and 6 to address the third objective.

To investigate the differences between the difficulties with the *x*- and the *y*-component of a vector, and to avoid any possible influence between the dot and cross product questions, following the methodology used by Barniol & Zavala [5], it was decided that half of the participants (population A) would solve Questions 1, 3 and 5 and that the other half (population B) would solve Questions 2, 4 and 6. The selection of these two populations was made randomly.

RESULTS AND DISCUSSION

This section is divided into three subsections addressing the three objectives of this study.

1. Misconceptions in the Vector Component Concept

In this subsection we investigate the misconceptions in the vector component concept. We analyze the difficulties with the *x*- and the *y*-component of a vector separately. At the beginning of this study we had a strong interest in understanding the causes of the most common wrong answer reported by Van Deventer [1]: to choose components with incorrect magnitudes. We wanted to understand in depth the general causes presented for this error: 1) misunderstandings of vector lengths and/or magnitudes, and 2) an unintentional incorrect judgment of length.

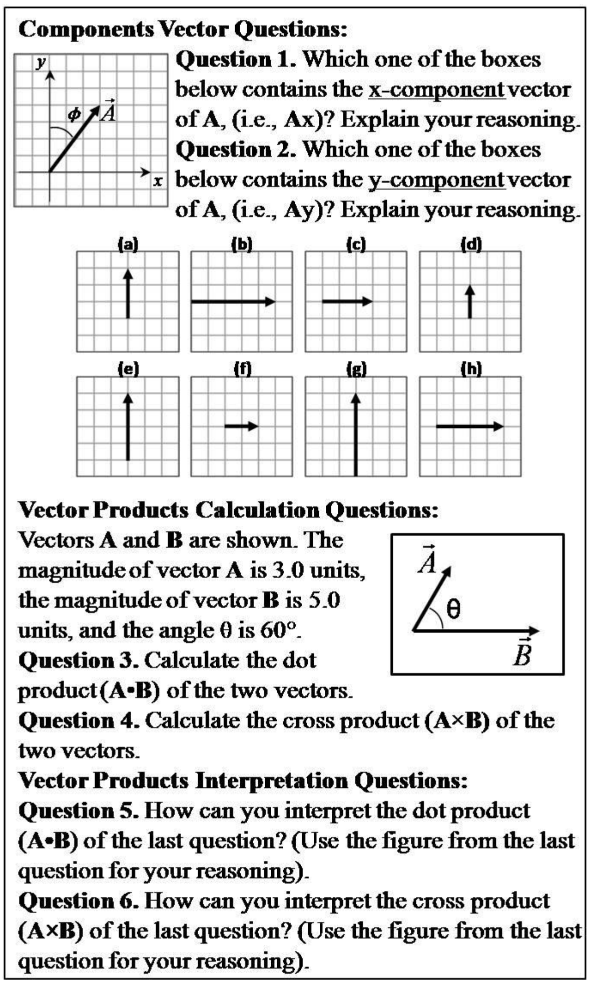

FIGURE 1. Complete set of questions administered to students in the study.

To investigate the causes of the most common errors and from results of a preliminary study (not shown here), we decided to modify Van Deventer's [1] questions by eliminating some distractors in which the component had been rotated, including distractors (b) and (d), and asking students for their reasoning. Table 1 presents the results of Questions 1 and 2.

TABLE 1. Results of Question 1 solved by population A and Question 2 solved by population B (N=409).

Responses	Question 1 *x*-comp.	Question 2 *y*-comp.
(a)	0%	10%
(b)	3%	0%
(c)	87%	5%
(d)	0%	2%
(e)	0%	80%
(f)	4%	0%
(g)	1%	3%
(h)	5%	0%

Table 1 shows that 13% of students have difficulties with graphical representation of the *x*-component (those who do not choose (c)). The most common error at 8% (options (b) and (h)) is to choose a component with a longer magnitude. Approximately half of these students state in their reasoning that the magnitude of the *x*-component of vector **A** must have the same magnitude as vector **A**. One example of a student reasoning is transcribed as follows: *"Because the size of the vector doesn´t change, if we move the vector towards the x-axis it would be option (b)."*

The second most common error, at 4%, is to choose a component with a shorter magnitude (option (f)). The greater part of these students reasoned as follows: *"The length of the vector has to be shorter and in the i-direction."* It seems that these students know the "rule" that the magnitude of a component of a vector must be shorter than the magnitude of the vector. However, they have difficulties identifying the exact graphical representation of the component.

Table 1 also shows that 20% of students have difficulties with graphical representation of the *y*-component (those who do not choose option (e)). The most common error at 12% (options (a) and (d)) is to choose a component with a shorter magnitude. Approximately half of these students justify their answers by stating again the "rule" that the component of a vector has to be shorter than the vector. As one student reasons: *"A component will not have the same magnitude as the resultant vector, it will be smaller, and it will not be half of it, just a little smaller."*

A smaller group, 3% of students, chooses a longer component (option (g)). Most of these students state in their reasoning that the magnitude of the *y*-component of vector **A** must have the same magnitude as vector **A**. A student writes: *"A_y is of the same magnitude as A, even as A_x."* This student also sketches a circular arc. It seems that he visualizes the components as a vector rotated toward each of the axes. Finally, it is interesting that 5% of students choose option (c), that is, the *x*-component. It seems that there is confusion due to the given angle, i.e., related to the *y*-axis. Some students who select this option relate in their reasoning the component chosen with the "opposite side" in the sine of the angle, which is usually used to calculate the magnitude of the *y*-component when the angle related to the *x*-axis is given.

2. Difficulties with the dot product

In this subsection, we analyze the difficulties in calculating the dot product (Question 3) and the misconceptions in the interpretation of this product (Question 5).

TABLE 2. Results of Question 3 solved by population A.

Responses	**%**
Correct	65%
Use equation $\lvert\mathbf{A}\rvert\lvert\mathbf{B}\rvert\sin\theta$	11%
Errors in the components of the vectors	8%
Correct result but with **i**-direction	3%
Multiplication of magnitudes	3%
Others	10%

Table 2 presents the results of Question 3. It shows that 65% of the students follow the correct procedure. The greater part of these students (92%) used directly the equation $\lvert\mathbf{A}\rvert\lvert\mathbf{B}\rvert\text{Cos}\theta$. However, a small part of them added the product of *x*-components and *y*-components of **A** and **B**. The most common error (11%) was to use $\lvert\mathbf{A}\rvert\lvert\mathbf{B}\rvert\text{Sin}\theta$ instead. Other difficulty students had, when trying to multiply components, is to make errors finding the components of the vectors (7%). The greater part of students with this difficulty had problems with vector **B**. Usually they calculated an *x*-component of **B** using a 60° angle. Finally, a small percentage (3%) adds incorrectly **i**-direction to the correct numerical result, and another small percentage (3%) multiplies directly the magnitudes of the vectors.

Question 5 allows us to investigate the misconceptions in the interpretation of the dot product. Table 3 shows the categories of the different interpretations of students. Categories were defined following the recommendations of Hernandez, Fernandez and Baptista [6]. Eleven percent of the students relate the dot product with the projection of a vector onto the other vector. These students give the basis of an adequate interpretation of this product. However, only 4% (of all students) mention explicitly in their rationale that the dot product is the projection of one vector onto a second vector multiplied by the magnitude of the second vector, giving a completely adequate interpretation. Also, 5% (of all students) interpret the dot product as "*the projection of one vector onto the other vector*" only and 2% (of all students) state in their reasoning that it is this projection plus the magnitude of the other vector.

TABLE 3. Categories of the interpretations of the dot product found in Question 5 solved by population A.

Categories	**%**
Projection of a vector onto the other vector	11%
Components multiplication	10%
Explaining the equation $\lvert\mathbf{A}\rvert\lvert\mathbf{B}\rvert\cos\theta$	14%
A vector	23%
The magnitude of a vector	9%
No clear explanation	16%
No answer	9%
Others	8%

As shown in Table 3, 10% of the students relate the dot product with components multiplication. Half of them, instead of talking about the "projection of vector **A** onto **B**", used the term "*x*-component of vector **A**." The other half of students interpreted the dot product as $A_xB_x+A_yB_y$. These students, and those who interpret the dot product by explaining the equation $\lvert\mathbf{A}\rvert\lvert\mathbf{B}\rvert\cos\theta$, are only using a definition instead of an interpretation, which indicates the low level of understanding of the concept.

The most common error (23%) is to interpret the dot product as a vector. Students sketch different vectors in their interpretations. Approximately one third of these students relate the dot product with the sum vector, another third with a bisector vector (a vector between **A** and **B**) and a small part with a vector in the positive horizontal direction. Van Deventer [1, 2] also found that students used the sum procedure when asked to calculate the dot product. Moreover, another common error (9%) is to interpret the dot product as the magnitude of a vector (usually the sum vector or a bisector vector again). If we add these two errors, we can establish that approximately one third of the population relates the dot product with a vector. Unfortunately, about 33% of students' answers did not permit us to establish incorrect ideas in their interpretation.

3. Difficulties with the cross product

In this subsection, we investigate the difficulties in calculating the cross product (Question 4) and the misconceptions in the interpretation of this product (Question 6). Table 4 presents the results of Question 4. It shows that 69% of students (the first three responses) follow the correct procedure to calculate the magnitude of the cross product. However, 32% of them do not identify a direction for the resultant vector. It seems that these students have problems with distinguishing the difference between the cross product and its magnitude. Furthermore, 31% (the first two responses) of students correctly calculate a vector although, 7% of them with an opposite direction.

TABLE 4. Results of Question 4 solved by population B.

Responses	**%**
Correct magnitude and direction	24%
Correct magnitude and opposite direction	7%
Correct magnitude, no direction given	32%
Use equation $|\mathbf{A}||\mathbf{B}|\cos\theta$	14%
Errors in the components of the vectors	7%
No answer	4%
Others	12%

In a similar way as detected in the dot product, students have difficulties using the adequate equation and calculating components. As shown in Table 4, 14% of students incorrectly use $|\mathbf{A}||\mathbf{B}|\cos\theta$ instead of the correct equation, as found by Van Deventer [1]. Also, 7% of students make errors finding the components of the vectors. Some students have difficulties with vector **B**; usually they calculate components of **B** using the 60° angle. Others have difficulties with the components of vector **A.** Students usually used the sine function instead of the cosine function (or vice versa).

Using the answers to Question 6, we investigated the misconceptions in the interpretation of the cross product. As in the last section, we defined categories (shown in Table 5) of the different interpretations.

TABLE 5. Categories of the interpretations of the cross product found in Question 6 solved by population B.

Categories	**%**
A vector, correct direction	22%
A vector, perpendicular to both vectors	8%
A vector, opposite direction	14%
Explain the equation used in the calculation	9%
Sum vector or bisector vector	8%
A “resultant” vector	5%
No answer	7%
No clearly explanation	19%
Others	8%

The students from the first and the third categories usually sketch vectors in their interpretations. Also, 30% of students (first and second categories) interpret adequately this product in a graphic way and 14% give the basis of an adequate interpretation of this product, but make an error with the direction of this vector.

Also, 8% of students interpret the cross product as a sum vector or a bisector vector (usually sketching these vectors). Van Deventer [1] also established that students used the sum procedure when calculating the cross product. Also, 5% use in their interpretation the term “resultant” vector which does not indicate anything, because it could be that they could be interpreting it as a sum or as the resultant of a vector product. These students and the rest in Table 4 make up about 38% of students whose answers did not permit us to establish the incorrect ideas in their interpretation. A further study is currently in process.

CONCLUSIONS

We found that, even after three introductory physics courses, students still have difficulties with vector components, in particular choosing answers with incorrect magnitudes. Some students think that the magnitude of a component is equal to the magnitude of the vector, and others know the “rule” that components are shorter than the vector, but have problems identifying the magnitude of the components graphically.

One of the most important findings of this study is that students finishing the introductory physics courses still have difficulties calculating products, related in some part to their problems with components. Also, they have serious difficulties in interpreting the scalar nature of the dot product and the vector nature of the cross product. A future study would try to understand whether students’ understanding of these products is dependent on the physical context.

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