Students' Difficulties with Quantum Measurement

Guangtian Zhu and Chandralekha Singh

Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA, 15260, USA

Abstract. We describe some common difficulties advanced undergraduate and graduate students have with concepts related to quantum measurement. We administered written tests to students enrolled in quantum mechanics courses and interviewed a subset of them to probe the difficulties in-depth and analyze their possible origins. Results from this research can be applied to develop learning tools to improve students' understanding of quantum measurement.

Keywords: quantum mechanics, quantum measurement, physics education research.
PACS: 01.40Fk,01.40.gb,01.40G-

INTRODUCTION

Quantum mechanics (QM) is challenging even for advanced students. There have been many investigations of the difficulties students have in learning QM [1-4]. Here, we discuss some of the findings of an investigation of students' difficulties with quantum measurement. The investigation involved administering written tests in QM classes and conducting in-depth individual interviews with a subset of undergraduate and graduate students at the University of Pittsburgh (Pitt) and other universities.

Quantum measurement formalism is quite different from classical mechanics, where the position and momentum of a particle evolve in a deterministic manner based upon the interactions. According to quantum theory, position, momentum and other observables are in general not well-defined for a given quantum state of a system. Moreover, the Copenhagen interpretation says that quantum measurement would instantaneously collapse the wavefunction (or the state of the system) to an eigenstate of the operator corresponding to the observable measured and the measured value is the corresponding eigenvalue.

The eigenvalue spectrum of an operator can either be discrete or continuous or a combination of the two. In an N dimensional Hilbert space, an operator $\hat{Q}$ corresponding to a physical observable Q with discrete eigenvalues has N eigenvalues q_n and corresponding eigenstates $|q_n\rangle$ where n is an index or set of indices. The state of the system at a given time t, $|\Psi(t)\rangle$, can be written as a linear superposition of a complete set of eigenstates $|q_n\rangle$. By projecting the state of the system $|\Psi(t)\rangle$ at time t onto an eigenstate $|q_n\rangle$ of the operator $\hat{Q}$, we can find the probability $|\langle q_n|\Psi(t)\rangle|^2$ of obtaining q_n when the observable Q is measured at time t.

After the measurement of the observable Q, the time-evolution of the state of the system, which is an eigenstate of $\hat{Q}$ right after the measurement, is governed by the Time-Dependent Schrödinger Equation (TDSE). Right after the measurement of energy, the state of the system is an energy eigenstate, and the probability density does not change with time since the only change in the wavefunction with time is an overall time-dependent phase factor. If the system is initially in an energy eigenstate at time t=0 and we measure an arbitrary physical observable Q after a time t, the probability of obtaining an eigenvalue q_n will be time-independent since the system was still in an energy eigenstate at time t at the instant the measurement of Q was performed. Therefore, the energy eigenstates are called the stationary states. On the other hand, measurement of position would collapse the system into a position eigenstate at the instant the measurement is made. The position eigenstate is a linear superposition of the energy eigenstates and the different energy eigenstates in the linear superposition will evolve with different time-dependent phase factors so that the probability density after the position measurement will change with time.

INVESTIGATION OF DIFFICULTIES

Our goal was to examine students' understanding of quantum measurement after traditional instruction. To simplify the mathematics and focus on the concepts related to measurement, we often used the model of a 1D infinite square well (e.g., the potential energy $V(x) = 0$ when $0 < x < a$ and $V(x) = +\infty$ elsewhere) during the investigation of difficulties. Below, we discuss some of the common difficulties.

1. Difficulty in distinguishing between eigenstates of operators corresponding to different observables

The measurement of a physical observable collapses the wavefunction of a quantum system into an eigenstate of the corresponding operator. Many students have difficulties in distinguishing between energy eigenstates and the eigenstates of operators corresponding to other observables. To investigate the pervasiveness of this difficulty in distinguishing between the eigenstates of operators corresponding to different physical observables, one of the multiple choice questions students were asked is the following.

- *Choose all of the following statements that are correct:*

(1) The stationary states refer to the eigenstates of any operator corresponding to a physical observable.

(2) If a system is in an eigenstate of any operator that corresponds to a physical observable, it stays in that state unless an external perturbation is applied.

(3) If a system is in an energy eigenstate at time t=0, it stays in the energy eigenstate unless an external perturbation is applied.

A. 1 only B. 3 only C. 1 and 3 only D. 2 and 3 only E. all of the above

The correct answer is *B*. In statement (1), the stationary states should refer to the energy eigenstates only. A complete set of eigenstates of an arbitrary operator $\hat{Q}$ cannot be stationary states if $\hat{Q}$ does not commute with the Hamiltonian operator $\hat{H}$. However, in a survey administered to over 200 undergraduate and graduate students, about 50% of the students mistakenly claimed that statement (1) is correct. In a junior-senior level QM class with only traditional instruction, none of the ten students selected the correct choice after lecture. Five out of the ten students thought that all three statements were correct because they had difficulty in differentiating between the related concepts of stationary states and eigenstates of other observables. Some students selected choice *A* which is interesting because one may expect that students who claimed statement (1) was correct and understood why a stationary state is called so may think that statement (2) is correct as well. In particular, for students who claimed statement (1) is correct, statement (2) may be considered "a system in a stationary state stays in that state unless an external perturbation is applied", which described the property of stationary states. However, students who selected choice *A* did not relate the stationary state with the special nature of the time evolution in that state.

2. Difficulty with possible outcomes of a measurement and the expectation value of the measurement result

The following multiple choice question was one of the questions administered to investigate students' understanding of the possible outcomes of a measurement for a given state of a particle in a 1D infinite square well when the measurement is performed. $\psi_1(x)$ and $\psi_2(x)$ are the ground state and first excited state wavefunctions.

- *An electron is in the state given by* $[\psi_1(x)+\psi_2(x)]/\sqrt{2}$. *Which one of the following outcomes could you obtain if you measure the energy of the electron?*

A. E_1+E_2 *B.* $(E_1+E_2)/2$ *C. Either* E_1 *or* E_2

D. Any of E_n *(n=1,2,3,…)*

E. Any value between E_1 *and* E_2

Because the energy eigenstates $|\psi_n\rangle$ are orthogonal to each other, $|\langle\psi_n|\Psi\rangle|^2=1/2$ for n=1 or n=2 and $|\langle\psi_n|\Psi\rangle|^2=0$ for all the other energy eigenstates E_n (n>2). Therefore, we can only obtain E_1 or E_2 with equal probability but no other energy. Only six out of fifteen students in a junior-senior QM class chose the correct answer *C* (either E_1 or E_2). The most common incorrect choice, selected by 27% of the students, was *B* ($(E_1+E_2)/2$) which actually represents the expectation value of energy. Students mistakenly claimed that the expectation value is the measured value of energy. The individual think-aloud interviews [5] indicated that many students were not only confused about the distinction between individual measurements and expectation values, but also had difficulty in distinguishing between the probability of measuring a particular value of an observable in a given state and the measured value or the expectation value. For example, during individual interviews, students often noted $\langle\psi_n|\hat{H}|\psi_n\rangle$ or even $\langle\Psi|\hat{H}|\Psi\rangle$ as the probability of measuring E_n in the state $|\Psi\rangle$. When these students were explicitly asked to compare their expressions for the probability of measuring a particular value of energy and the expectation value of energy, some students appeared concerned. They realized that these two concepts were different but they generally struggled to distinguish these concepts. They could not write an expression for the probability of measuring E_n either using the Dirac notation or in the position space using the integral form.

Also, some interviewed students had difficulty in connecting the probability of measuring each possible value and the expectation value of that observable in a given state. Since the expectation value in a given state can be approximated by the average of a large number of measurements of the physical observable on identically prepared systems, it is equal to the sum of the eigenvalues of the corresponding operator times their probabilities in the given state. Interviewed students often had difficulty with the statistical interpretation of the expectation value of the observable Q as the average of a large number of measurements on identically prepared systems in state $|\Psi\rangle$. When they were asked to calculate the expectation value of energy of a system, although they knew the probability for each possible outcome upon energy measurement, some of them tried to find the expectation value by sandwiching the Hamiltonian with the state of the system (i.e., $\langle\Psi|\hat{H}|\Psi\rangle$), then writing it in the integral form. These students often struggled with the calculation and made mistakes.

3. Difficulty with probability of measuring energy

When we explicitly asked students to find the probability of obtaining energy E_2 for the state $(|\psi_1\rangle+|\psi_2\rangle)/\sqrt{2}$ in a 1D infinite square well, many of them could provide the correct answer 1/2 by observing the coefficients. To evaluate whether students could calculate the probability of measuring a particular value of energy by projecting the state vector along the corresponding energy eigenstate for the case where the wave function is not written explicitly in terms of a linear superposition of energy eigenstates, the following question about a triangle shaped wavefunction in a 1D infinite square well was one of the questions students were asked:

- *The state of an electron in a 1D infinite square well at t=0 is given by $\Psi(x)=Ax$ when $0<x<a/2$, $\Psi(x)=A(a-x)$ when $a/2\le x<a$ and $\Psi(x)=0$ elsewhere. Here A is the normalization constant. What is the probability that an energy measurement at time t=0 yields E_2 ? (If there is an integral in your expression for the probability, you need not evaluate the integral but set it up properly with appropriate limits. Ignore the fact that the first derivative of the wavefunction is not continuous.)*

Unlike the wavefunction $[\psi_1(x)+\psi_2(x)]/\sqrt{2}$, which is composed of only two energy eigenfunctions, the triangle function state $|\Psi\rangle$ is a superposition of infinitely many energy eigenstates, i.e., $|\Psi\rangle=\sum_{n=1}^{\infty}c_n|\psi_n\rangle$. The expansion coefficient c_n equals $\langle\psi_n|\Psi\rangle=\int_{-\infty}^{+\infty}\psi_n^*(x)\Psi(x)dx$ and $|c_n|^2$ is the probability of obtaining E_n when energy is measured for the state $|\Psi\rangle$. Thus, to answer this question correctly, students must write $|\Psi\rangle$ as a linear superposition of $|\psi_n\rangle$ and find the component of $|\Psi\rangle$ along $|\psi_n\rangle$ (they could also write the probability in integral form as noted).

Only one student out of fifteen students in the undergraduate QM class provided the correct answer and some of them left this question blank. Others made two typical common mistakes. Twenty percent of them wrote down the energy expectation value $\langle\Psi|\hat{H}|\Psi\rangle$ to represent the energy measurement probability. In further interviews with some students, we asked how the expression $\langle\Psi|\hat{H}|\Psi\rangle$, which only involves state $|\Psi\rangle$, would favor energy E_2 over any other energy. Some of these students then changed their answers to $\langle\psi_2|\hat{H}|\Psi\rangle$, which was still incorrect. Another 27% claimed that the “probability” of measuring any physical observable was represented by $|\Psi(x)|^2$ according to the interpretation of wavefunction. These students were confusing the probability density for measuring position with the probability of measuring other physical observables such as energy.

To probe the pervasiveness of these difficulties, a similar multiple-choice question about a parabolic wavefunction was administered to 76 students in six universities as shown below:

- *Consider the following wavefunction for a 1D infinite square well: $\Psi(x)=Ax(a-x)$ for $0\le x\le a$ and $\Psi(x)=0$ otherwise. A is a normalization constant. Which one of the following expressions correctly represents the probability of measuring the energy E_n for the wavefunction $\Psi(x)$?*

A. $\left|\int_0^a\psi_n^*(x)\hat{H}\Psi(x)dx\right|^2$ *B.* $\left|\int_0^a\psi_n^*(x)\Psi(x)dx\right|^2$

C. $|\psi_n^*(x)\hat{H}\Psi(x)|^2$ *D.* $|\psi_n^*(x)\Psi(x)|^2$ *E.* $|\Psi(x)|^2$

Only 33% of the students chose the correct answer *B*. About 45% of the students incorrectly selected the distractor *A* which is an equivalent expression for $|\langle\psi_2|\hat{H}|\Psi\rangle|^2$. Another multiple choice question given to the same 76 students asked about the energy measurement outcome for the state $\sqrt{4/7}|\psi_1\rangle+\sqrt{3/7}|\psi_2\rangle$. Only 55% of the students provided the correct answer and 21% of the students incorrectly thought that other energies E_n besides E_1 and E_2 could also be obtained but the probability of measuring E_1 would be largest. Another 12% of the students thought that all of the possible energies E_n can be measured with the same probability.

4. Incorrectly believing that an operator acting on a state corresponds to a measurement of the corresponding observable

One of the questions on a survey given to more than 200 graduate students [3] asked them to consider the following statement:

- *By definition, the Hamiltonian acting on any allowed (possible) state of the system* $|\psi\rangle$ *will give the same state back, i.e.,* $\hat{H}|\psi\rangle = E|\psi\rangle$ *, where E is the energy of the system.*

Students were asked to explain why they agree or disagree with this statement. We wanted students to disagree with the statement and argue that it is only true if $|\psi\rangle$ is a stationary state. Eleven percent of the students answering this question incorrectly claimed that any statement involving a Hamiltonian operator acting on a state is a statement about the measurement of energy. Some of them who incorrectly claimed that $\hat{H}|\psi\rangle = E|\psi\rangle$ is a statement about energy measurement agreed with the statement while others disagreed. Those who disagreed often claimed that $\hat{H}|\psi\rangle = E_n|\psi_n\rangle$ because as soon as $\hat{H}$ acts on $|\psi\rangle$, the wavefunction will collapse into one of the stationary states $|\psi_n\rangle$ and the corresponding energy E_n will be measured. The following are two typical responses in this category:

* Disagree. The Hamiltonian acting on a state (measurement of energy) will return an energy eigenstate.

* When $|\psi\rangle$ is a superposition state and $\hat{H}$ acts on $|\psi\rangle$, $|\psi\rangle$ evolves to one of the $|\psi_n\rangle$ so we have $\hat{H}|\psi\rangle = E_n|\psi_n\rangle$.

Interviews and written reasonings suggest that these students believed that the measurement of any physical observable in a particular state is achieved by acting with the corresponding operator on the state. The incorrect notions expressed above are often over-generalizations of the fact that after the measurement of energy, the system is in a stationary state so $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$.

Individual interviews related to this question suggest that some students incorrectly believed that whenever an operator $\hat{Q}$ corresponding to a physical observable Q acts on any state $|\psi\rangle$, it will either yield an eigenvalue λ and the same state back, i.e., $\hat{Q}|\psi\rangle = \lambda|\psi\rangle$ or yield an eigenvalue λ_n with the corresponding eigenstates, i.e., $\hat{Q}|\psi\rangle = \lambda_n|\phi_n\rangle$. But the operator $\hat{Q}$ acting only on one of the eigenstates of $\hat{Q}$ should yield the eigenvalue and the same state back, i.e., $\hat{Q}|\phi_n\rangle = \lambda_n|\phi_n\rangle$. Since an operator acting on a state does not correspond to the measurement of the corresponding observable, acting on a general state $|\psi\rangle$ which is not an eigenstate of $\hat{Q}$ will not yield a single eigenvalue and the corresponding eigenstate.

SUMMARY

We find that students have many common difficulties with concepts related to quantum measurement. In particular, many students were unclear about the difference between energy eigenstates and eigenstates of other physical observables and what happens to the state of the system after the measurement of an observable. Students also had difficulty in distinguishing between the measured value, the probability of measuring it and the expectation value of the corresponding physical observable. They often did not think of the expectation value of an observable as an ensemble average of a large number of measurements on identically prepared systems but rather thought of it as a mathematical procedure where an operator is sandwiched between the same bra and ket states (the state of the system). These common difficulties should be taken into account while developing curricula and pedagogies to help students learn the formalism of quantum measurement.

ACKNOWLEDGMENTS

We thank NSF for financial support.

REFERENCES

1. P. Jolly, D. Zollman, S. Rebello and A. Dimitrova, Am. J. Phys. 66(1), 57-63 (1998).
2. M. Wittmann, R. Steinberg, E. Redish, Am. J. Phys. 70(3), 218-226 (2002)
3. C. Singh, Am. J. Phys. 76(3), 277-287 (2008)
4. G. Zhu and C. Singh, Am. J. Phys. 79(5), 499-508 (2011)
5. M. Chi, "Thinking Aloud", in *The Think Aloud Method: A Practical Guide to Modeling Cognitive Processes,*, edited by M. W. Van Someren, Y. F. Barnard, and J. A. C. Sandberg (Academic, London, 1994).