Student use of a material anchor for quantum wave functions

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Abstract. We explain the appropriate use of pipe cleaners to represent quantum wave functions in terms of material anchors. We then analyze the actions of one undergraduate quantum mechanics student in an oral exam situation with two related tasks, both involving the visualization of a 3-d structure to represent the real and imaginary parts of the wave function on one spatial coordinate. Instruction before the exam included several in-class activities involving building 3-d representations of wave functions for several potentials using pipe cleaners. Though the oral exam did not specify that students should or should not use pipe cleaners, the student in this analysis brought and used them successfully during the exam. Analysis of the students' use of this tool shows promise of benefit to future students in a more highly structured environment of instruction and assessment.

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INTRODUCTION

Quantum mechanics is regarded by most who encounter it as extremely abstract and difficult to understand. It is also traditionally presented mostly in algebraic form with expressions that are difficult to interpret in geometric or physical terms. Studies on student difficulties in quantum mechanics show that overall students are often at a loss to explain algebraic expressions conceptually or graphically [1,2]. In this article, we present a case study of how a high-performing student successfully used pipe cleaners as a cognitive tool to make sense of a quantum wave function on one spatial coordinate.

This article is also part of a sustained research project investigating student learning of complex numbers and functions, both in terms of real and imaginary parts as well as in terms of phase. Previous research has shown that many students have difficulty correctly applying and interpreting imaginary parts and phases [1,2,3].

Theoretical background: Material anchors

A *material anchor* [4] is a concrete object or system that serves as one input space to a conceptual blend [5]. In the blend, the material structure is infused with conceptual meaning, so that the conceptual constraints imposed on the blended cognitive system are apparent via the material structure of the anchor. Thus, certain conceptual stabilities are achieved, freeing the thinker(s) to reason about the implications of these stabilities rather than being bound in a cognitive state of effortful maintenance of the constraints. Algebra and 2-d diagrams traditionally serve as material anchors in cognition in quantum mechanics by stabilizing some ideas in order to allow others to flex.

Pipe cleaner as material anchor

A pipe cleaner is especially suitable as a material anchor for an energy eigenfunction in one dimension, *i.e.*, solutions to $\hat{H}\varphi(x) = E\varphi(x)$, and for instantaneous superpositions $\psi(x,t) = \sum c_n(t)\phi_n(x)$ of multiple such functions. The functions $\varphi(x)$ are, in general, complex-valued, and the quantities $Re[\varphi(x)]$, $Im[\varphi(x)]$, and x can correspond to a 3-d Cartesian coordinate system for the space containing the pipe cleaner; in that case, the pipe cleaner can be posed into a 3-d shape. This correspondence remains even for functions for which $Im[\varphi(x)] \equiv 0$; in this case, a pipe cleaner could be bent into a shape lying all in one plane (the $x, Re[\varphi(x)]$ plane) to indicate that the function is entirely real. The time dependence of energy eigenfunctions $\left(\varphi_n(x,t) = \varphi_n(x)e^{-\frac{iE_n t}{\hbar}}\right)$ is easily shown by rotating the pipe cleaner representing $\varphi_n(x)$ around the x-axis, from the positive real direction to the negative imaginary direction and so on.

The specific material feature of pipe cleaners that align well with quantum wave functions is a composite feature – its posability – that is, its ability to take a designed shape and hold it. This shape can be held while the pipe cleaner is rotated to show the time dependence of an energy eigenfunction. Pipe cleaners

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are also easy to superimpose; each one occupies a small space, leaving other space around it for other pipe cleaners, or components of the wave function. The posability that makes pipe cleaners suitable for representing energy eigenfunctions also limits the suitability of the pipe cleaner to represent a time-dependent superposition of energy eigenfunctions; in general, such a superposition will not hold a constant shape and would instead need to be animated in some way. Such superpositions may still be constructed, but only for a given instant during the time evolution of the superposition.

PHYSICS BACKGROUND

We analyze elements of one student's performance on two tasks. Below we review in algebraic terms the issues raised by the student in performing these tasks. ***Task 1:*** Explain whether the ground state energy eigenfunction of an infinite square well has an imaginary part as well as a real part. Also, explain how the ground state wave function evolves in time. In both cases, discuss the implications for representing the wave function graphically in three dimensions. ***Task 2:*** Draw the probability density function (of position) for a particular superposition of energy eigenstates for the infinite square well. Discuss how this function evolves over time.

Task 1 background

The infinite square well potential is zero for $0 < x < L$ and infinite otherwise. General solutions can be written as $\varphi(x) = Ae^{ikx} + Be^{-ikx}$ inside the well and $\varphi(x) \equiv 0$ outside the well. Applying the energy eigenvalue equation and boundary conditions yields the solutions $\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$. Given that $\sin kx = \frac{1}{2i}(e^{ikx} + e^{-ikx})$, an energy eigenfunction $\varphi_n(x)$ can be conceived as a linear superposition of momentum eigenfunctions, with equal and opposite momentum. Each of these energy eigenfunctions has no imaginary part, despite the fact that the momentum eigenfunctions that can be thought of as its components *do* have non-zero imaginary parts (see fig. 1). As time evolves, the wave function has time dependence $\varphi_n(x,t) = e^{\frac{-iE_n t}{\hbar}}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$, and thus has changing real and imaginary parts with constant magnitude $\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ at each position. The changing parts may be understood as resulting from a clockwise rotation of the function in the complex plane at a constant angular frequency $\frac{E_n}{\hbar}$.

FIGURE 1. Pipe cleaner models for the momentum eigenfunctions $\boldsymbol{e^{-ikx}}$ (green) and $\boldsymbol{e^{ikx}}$ (black), using a coordinate system in which the x-axis extends away from the viewer, and the positive imaginary axis ($\boldsymbol{Im}[\boldsymbol{\psi}]$) is one-quarter turn counter-clockwise from the positive real axis ($\boldsymbol{Re}[\boldsymbol{\psi}]$).

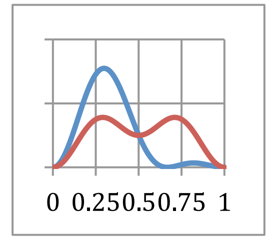

FIGURE 2. Normalized probability densities $|\langle \boldsymbol{x}|\boldsymbol{\psi}(\boldsymbol{t})\rangle|^2$ as functions of the dimensionless coordinate $\boldsymbol{x/L}$ for $t = 0$ (blue) and $t = t'$ (red).

Task 2 background

The quantum state $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|\varphi_1\rangle + |\varphi_2\rangle)$ is a superposition of the first two energy eigenfunctions of the infinite square well. The probability density as a function of position is shown in blue in fig. 2. This quantum state evolves in time according to: $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{-iE_1 t}{\hbar}}|\varphi_1\rangle + e^{\frac{-iE_2 t}{\hbar}}|\varphi_2\rangle\right)$

$$= \frac{1}{\sqrt{2}}e^{\frac{-iE_1 t}{\hbar}}\left(|\varphi_1\rangle + e^{\frac{-i(E_2-E_1)t}{\hbar}}|\varphi_2\rangle\right)$$

Since the exponential factor $e^{\frac{-i(E_2-E_1)t}{\hbar}}$ is a complex number of unit magnitude (like, for example, the number $-i$) and changing phase, there exists some time t' for which

$$|\psi(t')\rangle = \frac{1}{\sqrt{2}}e^{i\theta}(|\varphi_1\rangle - i|\varphi_2\rangle)$$

where $e^{i\theta}$ is an overall phase factor that depends on the value of t' but does not have any effect on observables like $|\langle x|\psi\rangle|^2$. The probability density at t' has reflection symmetry about $x = \frac{L}{2}$ and is shown in red in fig. 2.

Note that the wave function $\boldsymbol{\psi(x,0)}$ is entirely real, and could be sketched on a flat surface, while $\boldsymbol{\psi(x,t')}$ has non-zero real and imaginary parts. Though these parts could each be sketched on a flat surface, the whole function $\boldsymbol{\psi(x,t')}$ could not, despite the fact that it is single-valued. These wave functions and the energy eigenfunctions that compose them are shown with a pipe cleaner representation in fig. 3.

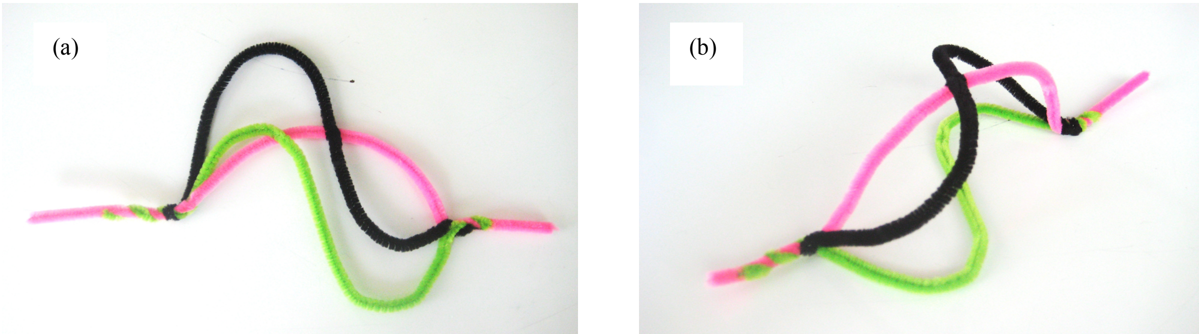

FIGURE 3. Pipe cleaner models of $\psi(x,0)$ (a) and $\psi(x,t')$ (b), with relative phase θ between the components of the wave function of 0 and $\frac{\pi}{2}$, respectively. In both pictures, black represents the overall wave function ψ, and pink and green represent the energy eigenfunction components that sum to ψ. All three pipe cleaners in (a) lie in one plane. Those in (b) do not; the pink and green lie in perpendicular planes, and the black curve does not lie in any plane.

DATA AND METHODS

The examples analyzed here come from a video record of the oral exam for one student (pseudonym Brooke) in a 1st semester upper division quantum mechanics course. The oral exam was the second of two in the course and occurred about 2/3 of the way through the semester. Instruction on wave functions included the use of the pipe cleaner as cognitive tool and addressed all of the basic shapes shown in figures 1-3. Students were not instructed or invited to bring pipe cleaners to the exam, but neither were they prohibited from doing so. It was expected that students would gesture 3-d shapes in the case that a wave function did not exist in one plane. (Most students demonstrated little ability to express such 3-d geometrical ideas with their hands on this exam.) Brooke was a high performer on the exam and in the course as a whole; she also chose to bring pipe cleaners to her exam (one of the 3 students out of the 7 analyzed). The entire exam was transcribed, with careful observation focused on how Brooke made use of pipe cleaners. Students were provided some general problem information several days before the exam, alerting them to the need to be able to represent one-dimensional wave functions in three dimensions. We focus on Brooke's responses to smaller scale tasks that emerged in the interaction during the exam.

ANALYSIS

We analyze two episodes involving Brooke's use of pipe cleaners, out of four such identified instances. We summarize some stretches of conversation, emphasizing Brooke's embodied interaction with the pipe cleaners.

Episode 1: Superposition of momentum eigenstates

Brooke draws the first energy eigenfunction for the infinite square well, $\varphi_1(x)$. The instructor asks whether the wave function at $t = 0$ lies purely in the plane of the board, or whether what's drawn is only a projection of a 3-d object onto the 2-d board. Brooke recalls the functional form of $\varphi_1(x,t)$, plugs in zero for t and sees that the complex exponential is 1.

B: So, I want to say that this is only in the plane of the board. But as time evolves… (stops talking, puts cap on marker) I need pipe cleaners.
I: OK. You can use these if you want. (Left in the examination room by another student)
B: I actually have some prepared. I think. (Assembles pipe cleaners into two oppositely-wound helixes, similar to fig. 1. Now seated at table, away from board) So I wanna say that… this is gonna be really abstract if I say it this way. Um. If we think of it more in the general solution here, (goes to the board) and if we think of it as two things that are wound opposite…(Gestures opposite twists with index fingers, pauses, goes to get pipe cleaners). It would be better with these. (Wraps pipe cleaner around water bottle to make a helix.)
So this moves around… (Index finger traces pipe cleaner helix) clockwise. (inaudible muttering) So if we think of it as a sum of these two oppositely wound… So the way that I'm puzzling this out is that, if I imagine an origin that runs directly in the center of both of these wound helixes (positions helixes to share an axis), and if I think of (points to board) the phi function, phi one, of x t, then I think of it as a vector sum that points from this origin (static pose of index finger pointing along axis of helix) to both of these helixes. And I can think that this picture here (phi one) should be a

representation of the way that these vector sums add together. So whenever you have a vector that's pointing straight out to this red one, at the same time you'll have a vector that's pointing straight out this way (points toward herself, perpendicular to helix axis), so you have one vector pointing that way and one vector pointing this way towards the white one at the same time, and this would be out here in this space (gestures oppositely directed, horizontal index fingers), so like out of the board (moves hand out of the board), they should cancel each other out, that's why we only see, why we graph this one curved line.

This episode demonstrates Brooke's understanding of the cylindrical nature of the superposition principle at work for wave functions in this representation. The first notable feature of this interaction is simply that Brooke elected to use the pipe cleaners during her explanation, as though she expected that they would improve the quality of her response. Second, as Brooke used them, several aspects of her behavior suggest that she was using them for her own cognitive benefit rather than for clarity of presentation to the instructor: her head was turned down toward her hands as she held, manipulated, and gestured around the pipe cleaners, her body was turned about 90 degrees away from the instructor, and her speech was often more quiet. Finally, Brooke used the pipe cleaners as an object of reference, both for her speech and gesture; she built shapes and then talked about and pointed to various aspects of their structure. It seems easy to imagine that the cognitive load would be greater, and in many cases too great, for a student to be able to trace those shapes in the air and then refer to those invisible structures with further speech and gesture.

Episode 2: Changing the relative phase of two different energy eigenstates

The instructor presents the initial quantum state of task 2 in algebraic form and asks Brooke to draw a probability density function for this state.

B: OK. So phi 1… I have pipe cleaners for this too. (Assembles two pipe cleaners like the pink and green of fig. 3a, with no third pipe cleaner to represent the sum) So it's going to have a peak over here and a dip over here. (Holds the pipe cleaners still with two hands and examines them.) Alright. So, the probability density. So I should have a larger one (peak) here and a smaller one over there. Something similar to this.

Brooke's function was similar to the black curve in fig. 2 but did not touch down to zero probability density at any location. Further discussion fixed this error, including a sketch on the board of the wave function, as in fig. 3(a).

I: Will there be a time at which this probability density function is symmetric about L over 2?
B: About L over 2. Yes, there will be a time.
I: When will that be? Or, what will the density function look like? And also when will that be?
B: Oh, when will it look like that actually? So this (pipe cleaners arranged as in fig. 3a) is the function we started at and I'm going to move it to here (repositions pipe cleaners to be in perpendicular planes, as in fig. 3(b)) and that should be symmetric, and that should be equivalent to plus pi over 2. (Quickly sketches the red curve from fig. 2 and proceeds to solve for t'.)

Here Brooke uses the pipe cleaners to make short work of the task, at times omitting outward signs of some steps in her thinking. For each time (0 and t'), Brooke uses the pipe cleaners to create the proper relative phase of the energy components of fig. 3. She does not review explicitly the superposition process she explained in task 1, but her explanation in task 1 suggests that she privately looked for a particular visual feature of her pipe cleaners, which is the (square of) the cylindrical radius from the x-axis to the imagined total wave function, and then sketched that on the board as the probability density function.

CONCLUSION

We have illustrated the substantive cognitive use of pipe cleaners by a high performing student in thinking about quantum wave functions. In future work, we plan to report on the effects on students in aggregate of similar, but more highly structured, tasks in instruction and assessment.

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