Finding Derivatives from an Equipotential Graph

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We examine how upper-division physics students find derivatives from an equipotential graph in an individual interview setting. We focus specifically on identifying the different kinds of behavior that students engage in when finding a derivative from an equipotential graph, and also on the representational elements that students use or introduce during their work. We find that the students were able to find the derivative successfully using a ratio-of-small-changes approach. Students engaged in behavior like sketching straight lines or arrows on the given graph as a way to choose points for calculating a ratio and to keep one variable constant for the derivative. We also saw students make sense of the equipotential graph and reinterpret the derivative they found using those representations using two other graphical representations: cross-sections and three-dimensional plastic surfaces. We find these results encouraging because the students had studied equipotential graphs and derivatives as ratios-of-small-changes as part of their junior-level electrostatics, suggesting that such a course is effective at helping students develop representational fluency for working with multivariable derivatives.

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I. INTRODUCTION

A common representation used in both introductory and upper-division physics courses is the equipotential graph (a contour graph where the contours represent the electric potential). As with most graphs in physics, one key way of understanding an equipotential graph is by looking at various rates of change (derivatives), some of which are components of the electric field (for an equipotential graph).

We are examining students' skills with finding derivatives from contour graphs as part of a larger investigation into student understanding of multivariable functions and derivatives [1–12]. Our specific research questions are:

1. What behaviors do students engage in when finding derivatives from an equipotential graph?
2. What representational elements do students attend to or invoke when finding derivatives from an equipotential graph?
3. What do these different behaviors and representational elements tell us about students' reasoning?

We address these questions using an interview task with upper-division physics students.

II. BACKGROUND AND METHODOLOGY

Research on how students understand graphs of functions of one variable is relatively robust [13–20], including topics like finding derivatives from graphs [21–29]. A common result from this research is that graphs can be difficult for students to interpret, especially graphs embedded in physical contexts. Similarly, how students understand partial derivatives has been studied in mathematics, chemistry, and physics contexts [7, 11, 29–45]. A consistent finding across this body of research is that students often have strong procedural abilities but struggle with more conceptual topics. On the other hand, how students understand multivariable graphs [46, 47] or interpret multivariable derivatives graphically [32, 33, 48–50] has not been studied as thoroughly.

To investigate how students find derivatives from equipotential graphs, we conducted individual interviews with seven physics students at the end of their junior year from the *Paradigms in Physics* program [51] at Oregon State University (OSU). This year consisted of intensive, interactive courses focused on problem-solving [52–54], including a course on electro- and magneto-statics (chapters 1, 2, and 5 of Griffiths [55]) that specifically includes contour/equipotential graphs (such as the one in Fig. 1), plastic surfaces (such as the one in Fig. 4b), and finding derivatives using ratios of small changes. These somewhat unique classroom experiences may impact the generalizability of our results.

The semi-structured interviews lasted about an hour and were video- and audio-recorded. The interview was divided into three phases: (1) an electrostatic potential prompt, (2) a thermodynamics prompt, and (3) questions about connections between the two prompts. In this paper, we focus only on phase 1, during which the students were given a graph (see Fig. 1) and the following (verbal) prompt:

Int.: "This graph shows an electric potential V. Determine the derivative of V w.r.t. y at the indicated point."

In designing this open-ended task [53], we chose not to include units, not to use the word partial derivative, and not to direct students to hold any variable constant so as not to limit the range of possible student responses. In an electrostatic context, it would be conventional to hold x constant when finding the derivative of V with respect to y. An expert solution to the task would likely be to choose two points along the $x = 6$ line, find both ΔV and Δy, and then divide them: $\frac{\Delta V}{\Delta y}$. We also provided a variety of tools, such as the plastic surface shown in Fig. 4, to make it easy for students to invoke tangible tools familiar from the classroom if they chose.

III. ANALYSIS

We used Thematic Analysis [56–58] to examine the data, guided by Zandieh's concept image framework for derivatives [59, 60]. With respect to research question 1, we identified three overarching behaviors (orienting to the graph, finding the derivative, and reflecting). Examples of student work for each theme is presented in the following subsections, along with a description of relevant subthemes (representational elements identified or invoked by the students).

A. Orienting to the graph of *V*

Most students began by orienting to the given graph (see Fig. 1). This graph represents a relationship between three variables: V, x, and y. Six of the seven students verbally

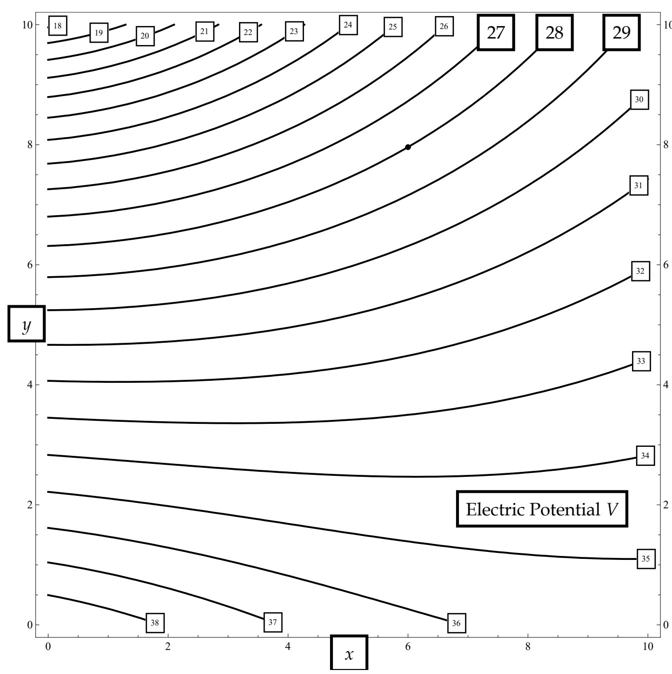


FIG. 1: An equipotential graph (some text enlarged for readability); students were asked to find the derivative of V with respect to y at the indicated point.

identified the labels on the x- and y-axes. Similarly, six students identified the contours as conveying some information about V: either the explicit values of V, the fact that V is constant along the lines, or the sense of increase given by the numbered labels on the graph. For example, after identifying x and y, Alex pointed at a contour and said:

Alex: "And then these lines are supposed to represent a two-dimensional V."

Three students questioned the meaning of the boxed numbers labeling each contour, as in the example below:

Lee: "Are these like the numbered values of V?"

Here, Lee asked if the numbers should be interpreted as values of the potential, rather than stating that they are values of the potential. No student asked for the meaning of the tick marks on the axes to be clarified.

Four students wrote the target derivative using either $\frac{\partial V}{\partial y}$ or $\frac{dV}{dy}$. No students indicated symbolically that any variable should be held constant.

At some point during Phase 1, all seven students gave a verbal description for the derivative such as:

Pat: "How much the graph is changing in just the y-direction."

This language (which does not mention a rate of change) matches the *change* interpretation we identified in a previous study [6]. Furthermore, the term "just" suggests that Pat meant to consider a change in the y-direction involving no change in the x-direction.

B. Finding the derivative

We now discuss three different methods students used to find a derivative: the ratio-of-small-changes method, the along-a-contour-line method, or the gradient method. The ratio-of-small-changes method was the most common (in fact, all seven students eventually did this with no prompting from the interviewer). Their strategy correctly involved choosing two points, separated vertically. However, some students chose the indicated point and a point on a neighboring contour while others chose points on the neighboring contours in each direction. A few students even tried more than one pair of points to check whether or not their answer was reasonable.

The students used both graphical and verbal representations to signal which derivative they determined. Six students drew a (vertical) line, arrow, or line segment intersecting the chosen points. This suggests that graphical thinking was particularly useful in solving this task, which seems reasonable since the given representation is also graphical. Four students used the phrase "in the y-direction" or "with respect to the y-direction" to refer to the derivative, a verbal representation with graphical overtones we have previous identified [6]. Interestingly, only three students verbally specified that they were holding x constant, even though the term "constant" is canonical mathematics language for partial derivatives, and only one student used the term "partial derivative."

Students then went on to find differences or changes in the values of V and y between the chosen points and divide those changes to find a derivative. Six students explicitly wrote the derivative as $\frac{\Delta V}{\Delta y}$. All of the students who wrote $\frac{\partial V}{\partial y}$ or $\frac{dV}{dy}$ when orienting to the graph also wrote $\frac{\Delta V}{\Delta y}$ when finding the derivative. This suggests a strong connection between these symbolic representations for the students.

The second method for finding the derivative (the along-a-contour-line method) was unique to Pat, who initially chose points along the $V = 28$ contour line and measured y-coordinates and Δy (shown in black on the right side of Fig. 2). Having found Δy, Pat contemplated how to find ΔV, writing $f(x, y) = 28$, $dy = 0.5$, and df. Pat stared at this last expression for a few moments, smiled, and crossed out this work (see Fig. 2), eventually coming to an epiphany:

Pat: "I think I'm misremembering how the plots work. Cause I know that this line is [writing] $f(x, y) = 28$ so this is, oh! I'm remembering it correctly, so I'm not looking at the individual curves, I'm supposed to be looking at how the whole field changes at that point. So at this point let me just draw a line through it [draws a horizontal and a vertical line in green]. [...] So what I was doing before was I just looking at the single, just the one line [traces the line], but that's not what I'm supposed to do when I'm looking at how the function as a whole is changing in one direction."

Pat then used a difference quotient to evaluate the derivative numerically at the point (shown in green in Fig. 2).

Pat was the only student to try to find a derivative along one of the constant-V contour lines. This is not an unreasonable thing to try, especially given that derivatives of single-variable functions are typically found in exactly this way. Pat's labeling of the contour line as $f(x, y) = 28$ (the second time) appears to have triggered an understanding that the potential, which is constant along any one line, only changes *between* contour lines. This information about the contour graph is implicit, but the added symbolic representation seemed to help

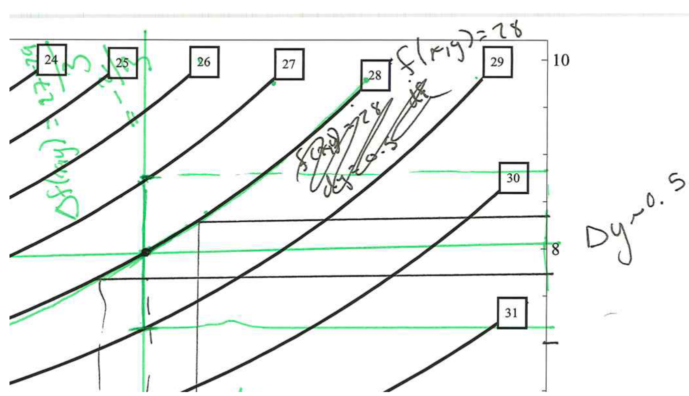
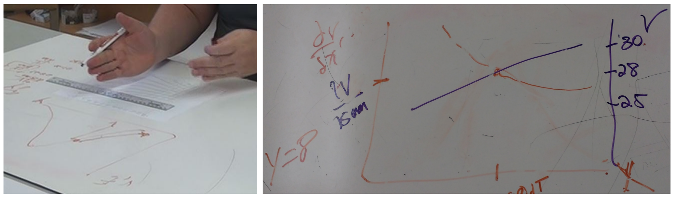


FIG. 2: Pat is initially finding the derivative along the $V = 28$ contour line (work in black), then switches to finding the derivative along a vertical path (work in green).



(a) (b)

FIG. 3: Alex's work: (a) an initial mock cross-section of V and (b) a more precise cross-section of V and its derivative.

(a) (b)

FIG. 4: Drew (a) indicates how the contour graph can be viewed as a surface and (b) interprets the derivative of V as the slope of a tangent line on a plastic surface.

Pat make it explicit, which in turn led Pat to reframe the task and eventually find the requested derivative along a vertical line of constant x.

We would like to call attention to some interesting word choice on Pat's part. Early in the excerpt, Pat goes from talking about "individual curves" to looking at "the whole field." This may indicate that Pat has gone from thinking about the given graph as a collection of functions of one variable to viewing the entire graph as a single *field*, a function of two (spatial) variables. Interestingly, Pat later uses the term "function," so it is not clear to us whether Pat views terms like "field" and "function" as meaning identical or distinct things.

The last method we discuss is the gradient method. Only two students tried this; both began by sketching the gradient vector and then considering how to proceed. Ultimately, neither student was able to find the derivative of V with respect to y using this method, and both eventually switched to the ratio-of-small-changes method.

C. Reflecting on the derivative of *V* with respect to *y*

After finding a numerical value for the derivative, it was common for students to reflect in some way on their answer. For example, four students assessed whether or not the sign of their answer was reasonable, a common physics sensemaking strategy [61], including one student who self-corrected a sign error. Some reflection was spontaneous, while some was in response to questions from the interviewer asking students to restate their procedure for finding the derivative, to explain interesting aspects of their overall responses, or to give a physical interpretation for the derivative. In some cases, the reflections provided us with new insight into how students found the derivative. In particular, several students reinterpreted aspects of the given graph and the target derivative using an alternate graphical representation.

We begin by discussing Alex's reflection after finding the derivative along the vertical path. Aside from reversing x and y, Alex initially had no trouble approximating the derivative as $\frac{\Delta V}{\Delta y}$, finding the changes, and dividing them. Alex referred to the derivative as a "gradient" throughout the interview, but did not otherwise use any symbols or language appropriate to the vector quantity known as the gradient. After Alex's calculation, the interviewer asked Alex to give a physical interpretation of the derivative. Alex sketched a mock graph of V $vs.$ y as a cross-section of a hill (see Fig. 3a), accompanied with a hand gesture to indicate a cross-section. Alex described how this new sketch relates to the given graph:

Alex: "I mean right now I'm just doing linearly, cause that's what you, the gradient [points to $\frac{\partial V}{\partial y}$] is linear, we're keeping x constant right now."

Int.: "Can you say more about that?"

Alex: "Well you wanted the change in potential with respect to y [points at derivative again]. So in order for me to do that I made x constant and did a line straight across."

We clarify that Alex used the word "linear" to mean "straight" rather than "varying linearly," which aligns with the language Alex used at the end of the excerpt.

After this, Alex noticed they had mixed up x and y. Alex elected to change the derivative under consideration to be $\frac{\partial V}{\partial x}$ rather than recalculate the derivative with respect to y. They made a new graph (the orange curve in Fig. 3b) showing the derivative $\frac{\partial V}{\partial x}$ at $y = 8$. In response to a request from the interviewer, Alex added a sketch of V to the same set of axes (the purple curve in Fig. 3b).

Alex coordinated two different kinds of graphs to help express the idea of holding a variable constant when finding the derivative. Two other students sketched a one-dimensional graph similar to Alex's "cross-section." We have previously observed students interpreting a multivariable derivative in this way [6], but in that study it was much rarer than other interpretations. The fact that several of the interviewees referenced a cross-section suggests that it is a familiar idea that many students might be able to tap into. Furthermore, students' use of cross-sections in an explanatory way indicates a potential to form and strengthen connections between ideas in different representations.

Three students used a plastic surface to interpret the derivative as the slope of a line tangent to the surface. (The students had used the surface, made available by the interviewer at the start of the interview, in their electrostatics course.) For example, Drew stated that the derivative "would be the slope of a tangent line on the surface there." Earlier in the interview, Drew imagined a surface corresponding to the given contour graph, as shown in the gesture in Fig. 4a. Then, Drew used a finger placed near the plastic surface to represent the tangent line (see Fig. 4b), clarifying:

Drew: "So I think what I'm trying to say here is that if you, at some x value here [points to plastic surface], and you put something that sits tangent to the surface at this point here [points to contour graph], then the slope of that line [tilts finger] in the y-direction is $-4/3$."

For the students who sketched a one-dimensional cross-section graph or made use of a plastic surface, the alternative representation appeared to help describe not only how the three variables (V, x, and y) are related, but also what the derivative is. The students began by describing the new representation and then used it to give some meaning to the derivative that is not possible with the contour graph alone (*e.g.,* the slope of a tangent line).

IV. DISCUSSION

There is a strong alignment between our research questions and the instructional objectives of the junior-level physics courses that the interviewees completed. Pedagogically, these students' instructors want students to have a large and diverse toolbox of representations, to be flexible in their ability to use those representations, and to be able to choose representations appropriate to the task. Much of upper-level physics (and beyond) involves situations where the exact functional relationships may not be known, requiring physicists to be able to extract physically important quantities and relationships from other sources, such as graphical data. Furthermore, we have previously noticed in the classroom that students entering junior-level physics typically need some (though not necessarily a large amount of) instruction on how to deal with contour graphs and their associated derivatives, suggesting our students did not develop such skills in their introductory math and physics courses. We are therefore particularly encouraged to see that the interviewees were highly successful at solving the given task and that the students did in fact demonstrate an understanding of and a flexibility with different representations. This suggests that the activities completed by the students [62] are indeed promoting students' representational fluency.

We now discuss the themes in the context of our research questions. First, we found students engage in three overarching behaviors when finding derivatives from an equipotential graph: (1) orienting to the graph, (2) finding a ratio of small changes, and (3) reflecting on their process. Second, students invoked a variety of representational elements, including vertical lines, Leibniz notation, and alternate graphs like cross-sections. We focus especially on interpreting our results in the context of our third research question: "What do different behaviors and representational elements tell us about student reasoning?" For example, the initial orienting behavior is evidence for the students having a strong understanding of the relevant underlying concepts, as they appeared to be thinking carefully about the graph rather than performing an algorithmic calculation. Using a ratio of small changes to calculate the derivative suggests the students were capable of finding derivatives numerically and graphically (not just *symbolically*). We note that our interviewees took an electrostatics course that addressed a ratio-of-small-changes interpretation of the derivative, and that this particular result may not generalize to students with different learning experiences. Most powerfully, the ways in which students reflected on and reinterpreted the derivative (using cross-sections or surfaces) demonstrated graphical reasoning skills that formed an invaluable part of students' sensemaking. Such use of multiple representations, which has been studied in a variety of ways in the PER community, is often highly prized in physics classrooms [61, 63–71].

The intersection of representational elements and student reasoning is more complex. Since we gave the students a contour graph, it is unsurprising that many of our observations involve graphical representations. Most commonly, nearly all students drew a vertical line, arrow, or line segment on the graph—which we interpret as a powerful (if simple) tool to support students' reasoning and/or calculation. Of particular note is that this representational element helped students hold an appropriate variable constant while finding the derivative, possibly accidentally. For example, four students used language like finding the derivative "in the y-direction," which invokes the vertical direction without explicitly using the word "constant." We cannot say for certain whether or not these students were aware they had found a derivative "with x held constant." On the other hand, three students used these words explicitly (the prompt intentionally did not include this language). Furthermore, no students used any symbolic notation to indicate what was held constant, such as $\left(\frac{\partial V}{\partial y}\right)_x$, though most wrote $\frac{dV}{dy}$, $\frac{\partial V}{\partial y}$, or $\frac{\Delta V}{\Delta y}$ at some point during the interview. Although subscript notation is uncommon in electrostatic contexts, the students had previously completed a thermodynamics course where this notation was used.

Our findings have interesting implications when considered alongside similar tasks in other contexts. As discussed in Sec. II, our interviewees were asked to solve a more complex *thermodynamics* derivatives task based on a contour graph after solving the electrostatic task discussed in this article. Despite many similarities between the tasks, students struggled profoundly both with finding a derivative and with using and interpreting different representational elements. We provide a thorough analysis of student performance on that task and its implications in Ref. [12].

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