

**Investigating and Improving Student Understanding of Quantum Mechanics Using  
Research-Validated Clicker Question Sequences and Tutorials**

by

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# **Investigating and Improving Student Understanding of Quantum Mechanics Using Research-Validated Clicker Question Sequences and Tutorials**

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University of Pittsburgh, 2024

Quantum mechanics is notoriously challenging, and research has found that students struggle with many common difficulties when learning it. It is also proving to be a critical piece of many exciting fields that are all but assured to see great development and expansion in the coming years; the Second Quantum Revolution is upon us. Quantum information science and engineering is a rapidly unfolding field, requiring talent from many disciplines, that aims to leverage the potential of quantum systems for many practical applications. To prepare students for the opportunities afforded by these advances, a strong foundation in quantum mechanics is essential. The work that I present in this dissertation is focused on helping students achieve this understanding. By investigating the common difficulties that students have in key concepts related to quantum mechanics and quantum computing, a guiding framework can be established and followed to develop research-validated, active-engagement instructional tools. These tools include Clicker Question Sequences (CQSs) on (1) the basics of two-state quantum systems, and changing basis in two-state systems; (2) time-development of two-state systems; (3) quantum measurement of two-state systems, and (4) measurement uncertainty in two-state systems. In addition to these, I have developed and validated Quantum Interactive Learning Tutorials (QuILTs) consisting of guided-inquiry teaching-learning sequences for (1) the Bloch sphere and (2) the basics of quantum computing. In each case, cognitive task analysis from both expert and student perspectives was either carried out directly or built upon from the results of prior investigations. I discuss the results

of implementations of these learning tools in authentic classroom environments, which involves both online and in-person administrations and multiple instructors. In each case, student performance after engaging with the learning tools increased noticeably, and dramatically for some difficult concepts.

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## Preface

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## **1.0 Introduction**

One major goal of physics courses is to help students develop a deep conceptual understanding and appreciation of the course content, as well as build a suite of strong metacognitive, reasoning, and problem-solving skills. While many introductory physics courses across the United States have begun trending towards the use of research-validated learning tools that promote active engagement, such tools remain exceptionally useful (if currently underutilized) for upper-division courses as well, including quantum mechanics (QM). Prior research suggests that students in QM courses often share common difficulties, but that research-validated learning tools can effectively help students develop a functional understanding [1–26]. Furthermore, the importance of QM instruction is underscored by the onset of the Second Quantum Revolution, in which new problems and solutions are being devised by utilizing the potential of quantum information and quantum computing principles. It is the prerogative of physics departments to prepare interested students well for such environments and opportunities [27–32]. Our research group at the University of Pittsburgh has been involved in many studies involving QM instruction and the development of learning tools to help students develop a robust understanding [1,3,11,12,27–29,33–84].

## **1.1 Research Frameworks**

Available class time is a pointedly limited resource in efforts to engage students in the learning process and strengthen student learning outcomes. This is particularly true for QM courses,

in which the content can often be counterintuitive and appear inconsistent with students' prior experiences in everyday life as well as classical physics courses. Instructors, therefore, must consider research-based pedagogical approaches to engage students with these challenging concepts. Several research-based pedagogical approaches were used in the development of these instructional materials. They include the balancing of innovation and efficiency [85], leveraging of peer interaction-driven collaborative learning [86,87], and the Interactive-Constructive-Active-Passive (ICAP) framework [88].

### **1.1.1 Innovation and Efficiency**

Schwartz et al.'s Preparation for Future Learning (PFL) framework [85] emphasizes the importance of both innovation and efficiency in instruction. One interpretation of these concepts has innovation referring to developing students' ability to transfer existing knowledge to novel situations, such as through exploratory labs, and efficiency indicating approaches that communicate large quantities of information to many students at a time, as in lecture-based instruction. Schwartz et al. observe that, in most traditional classrooms, efficiency is typically emphasized at the expense of innovation, leading to "routine experts" who are fluent in a specific type of task but struggle with transferring their learning to new situations. At the same time, an exclusive focus on innovation can result in the "frustrated novice" who is unable to make any meaningful progress on problems without the guidance and scaffolding support of more structured instruction. Thus, they suggest that balancing innovation and efficiency in learning activities by way of an "optimal adaptability corridor" can improve students' conceptual understanding and transfer of knowledge to new contexts; these "adaptive experts" are able to complete a wide variety of tasks requiring transfer of knowledge in their area of expertise with speed and accuracy [85].

Nokes-Malach and Mestre built upon this work and proposed that allowing students to struggle through innovative tasks imparts a “hidden” effectiveness that ultimately primes students to be more mindful and aware of their knowledge gaps compared to when they receive efficient direct instruction [89]. Integrating collaborative learning and whole-class discussions when incorporating research-based CQSs into a traditional lecture-based course offers one mechanism for balancing efficiency and innovation in instructional design. This approach can provide students opportunities to practice productive struggle while still enjoying the benefits of guidance from the instructor’s lectures and discussions before and after the collaborative activities.

### **1.1.2 Collaborative Learning**

Collaborative learning can be productive in physics classrooms [86,87], particularly when individual accountability and positive interdependence have been suitably incentivized. Though students may not be at the same point in their learning of a particular concept, they may still have a better understanding of one another’s difficulties than the instructor, allowing them to more effectively and efficiently clarify issues related to those difficulties when working together [90,91]. Additionally, explaining their thoughts to one another demands full engagement and analysis of their own thought processes in order to effectively communicate, so efforts to explicitly present their knowledge give clarity to both the explainer and the listener (roles which, of course, can fluidly change). The fine-grained parsing of each member’s relevant knowledge may lead to more careful consideration of the problem and even lead to co-construction of knowledge, yielding insights that were not obvious beforehand [92]. Collaborative learning also often creates inherent expectations of accountability for each student, and it has been suggested that students are often more comfortable interacting with peers about their difficulties than with an authority figure [93];

moreover, through practice in collaboration, students not only can develop more comfort and competencies in their communication skills [94–98], but may also positively influence their self-efficacy and other motivational beliefs, which are associated with better student performance and persistence in STEM courses [99,100]. Furthermore, in-class collaboration can forge connections between students that they can continue to use as resources outside of class, for example, when completing homework assignments.

Peer collaboration has been shown to be an effective method for students to learn in previous work for a variety of contexts [87,101], including in physics [102]. Students' performance on conceptual physics questions can receive a substantial boost from working in pairs compared to when only working individually, even if only one or neither student knew the correct answer for any given question. In particular, co-construction allowed student pairs whose members did not find the correct answer individually to generate it approximately 30% of the time, and retention was strong, as the effects persisted when the students were assessed again individually [103]. There is also some evidence that these effects may apply to QM concepts as well [83].

### **1.1.3 ICAP Framework**

Emphasizing the importance of collaborative learning, Chi et al. proposed the ICAP framework, in which there are four broad modes of learning: Interactive, Constructive, Active, and Passive (ICAP). The passive mode, generally associated with the weakest learning outcomes, comprises methods of traditional and direct instruction, such as through classroom lectures, readings, or videos, in which students only engage through visual and auditory channels without any further activity. The active mode, distinct from the broader term “active learning,” is used in this framework to describe situations in which students are making choices or physical

manipulations in their learning materials, but without generating explicit new knowledge or connections to old knowledge. Such generation of knowledge is a hallmark of the constructive mode, in which students are reinforcing and strengthening their own knowledge structure through reflection and metacognition, which can be encouraged through a variety of activities. Finally, the interactive mode is characterized by co-construction in small groups and is associated with the largest improvements in student retention and performance as compared to the passive mode. As suggested by this description, the interactive mode is only possible when all constituent students are already situated and working together in the constructive mode. Broadly speaking, research shows that each successive mode subsumes the behaviors and benefits of all the modes that rank beneath it, in the descending order I/C/A/P [88,104].

#### **1.1.4 Peer Instruction**

The frameworks described in the preceding sections are all leveraged by clicker questions, popularized by Eric Mazur using his *Peer Instruction* method [86]. In this method, conceptual multiple-choice questions (the “clicker questions”) are presented to a class, such as in slideshow format, for students to consider and answer. When students answer anonymously and are given grade incentives for both participation and correctness, with more emphasis on the former, they are likely to put in good-faith effort to answer the questions to the best of their ability without experiencing the debilitating effects of high stress. When students are allowed to discuss with one another in small groups to reach a consensus answer for which they receive immediate feedback, the true strength of peer instruction is revealed as they can engage in powerful co-constructive processes that allow them to converge upon the correct answer much more frequently than they could individually. Clicker questions have been widely adopted, with proven effectiveness and a

low barrier of entry, as they fit in relatively agreeably to typical lecture-based courses [105]. In this research, clicker questions were used as a supplement to traditional lecture-based instruction, giving students a second opportunity to engage with the material more deeply through discussions with one another, the instructor, and the whole class. While these questions can be successfully implemented without additional technological tools, this research opted to use clickers, small devices that make up an electronic response system which automatically tracks student responses in real time. During online implementations, this system was readily convertible to the environment and tools of the Zoom remote meeting software.

## **1.2 Development of clicker question sequences (CQS)**

One of the two main parts of the work chronicled in this thesis concerns the Clicker Question Sequence (CQS). When clicker questions are validated by research and presented in sequences with increasing complexity to develop a particular concept, they can systematically help students with a particular theme that they may be struggling with. Previously, such CQSs have been successfully developed, validated and implemented on several key topics in QM [69,71–73,106]. A CQS can compactly present many difficulties, related ideas, and extensions of earlier content all at once to students in an organic and manageable way. Furthermore, an instructor can nimbly and responsively vary the pace of the material to quickly review things that students are proficient in and focus on concepts that they find more difficult, such that CQSs can be finished in class in a reasonable amount of time despite the relatively large number of questions.

Previous studies have investigated the effectiveness of CQSs in topics such as addition of angular momentum [69] and Larmor precession [73], and I build on this work with four additional CQSs on the following concepts, specifically as they pertain to two-state quantum systems:

1. Change of Basis and Basics of Quantum Systems
2. Quantum Measurement
3. Time-development
4. Measurement Uncertainty and Uncertainty Principle

The focus on two-state quantum systems reflects an acknowledgement of the meteoric rise of Quantum Information Science and Engineering (QISE), an exciting interdisciplinary field whose practitioners must be very familiar with ideas regarding the quantum bit (qubit), a two-state quantum system that serves as one of the foundations of quantum computing. Furthermore, two-state systems are regarded in their own right as critical knowledge in QM, both with wide applications in areas of active research and as a suitable way to introduce students to QM altogether (known as “spins-first,” an approach favored by J. J. Sakurai’s influential graduate text *Modern Quantum Mechanics*). While these CQSs are framed in terms of spin- $\frac{1}{2}$  systems as typically taught in a quantum physics course, they are readily adaptable for use in a quantum computing-centered course as well, as the targeted concepts are broadly applicable and useful in both fields.

The CQS questions are designed to be a collaborative activity involving peer and whole-class discussions, requiring students to think about conceptual knowledge and their implications in new and different ways. This satisfies the criteria for an interactive activity under the ICAP framework. During the online implementation, however, constraints imposed by time and the affordances of the technology lessened the feasibility of promoting the small-group-discussion component. Instead, students were simply given the questions to think about, and they answered

the questions via individual polling. Therefore, using the ICAP framework, the students are in the constructive learning mode while engaging with the CQS content in the online year, and in the interactive mode during the in-person years. I outline the development and validation process below and in the next chapters, with further details also available in Ref. [107]. I then report on the results of the in-class implementation of the four CQSs listed above. For each of the four broad curriculum topics, student understanding was assessed with a pre-test and post-test developed and validated alongside the CQS itself. In each case, there was obvious improvement from the pre-test to the post-test, attributable at least in part to the effects of the CQSs.

### **1.2.1 Development and Validation Details**

The CQSs in this suite of topics pertinent to two-state systems took inspiration from the running start provided by earlier CQSs and QuILTs on similar concepts presented in different contexts (e.g., wavefunctions). This allowed for a faster, more streamlined development trajectory than would have otherwise been possible. Starting from a number of questions and concepts that had already been found, through previously-conducted faculty discussions and student interviews, to be common sources of student difficulty, I constructed four distinct CQSs that investigated these concepts, with common and compelling difficulties embedded in distractor choices and suitable breaks for class discussion to address points of likely cognitive conflict. The process continued, engaging the primary researchers in multiple rounds of discussion and iteration, and the CQSs were then reviewed by several faculty members who had many years in teaching courses in QM and related fields, such as solid-state physics. The CQSs were also subjected to student interviews to ascertain whether students were able to comprehend and reason through the questions as

intended. The CQs continued to receive minor adjustments throughout the years of in-class implementation.

### **1.2.2 Implementation**

The CQs were implemented across multiple years, with an unusual opportunity to investigate their effectiveness in a remote learning environment as a result of the COVID-19 pandemic that necessitated a lockdown of nearly all public facilities nationwide and across the world. While additional factors and stressors are undoubtedly present in a major public health emergency, and I do not mean to suggest that these conditions are reflective of any given online-based learning environment, the circumstances did allow the CQs to be used in a synchronous conferencing software-based virtual classroom in a way that has not been extensively studied. While the timings and presentations of the pre-test, CQ, and post-test were largely the same, the Peer Instruction feature could not be implemented and was forgone, and students took the pre-test and post-test at the same designated time but in less controlled conditions; for example, students had the option to turn their cameras off, and all students exercised this option.

All CQs were also implemented in at least one in-person class, in some cases two, with a familiar structure.

## **1.3 Development of tutorials**

Considering the rising importance of QISE as we enter the Second Quantum Revolution [28,29], it behooves modern physics education to provide students with a strong foundation in

avenues toward future QISE careers. The University of Pittsburgh, for example, has approached this by offering a Quantum Computing and Quantum Information Certificate, a program that students of various fields including chemistry, computer science, etc. can opt to pursue. The course that all students are required to take to complete this certificate is Foundations to Quantum Computing and Quantum Information, a course aimed at a more general audience than typical physics QM courses that uses language and structures relevant to QISE. This was the type of course for which I developed and validated two tutorials, though in keeping with the same philosophy of adaptability applied to the CQs, these tutorials can be and were used in both the QISE and quantum physics contexts. The tutorials followed a development and validation protocol similar to that of the CQs, going through rounds of iteration with multiple faculty members and student interviews. The main difference is that the majority of the content regarding quantum computing was conceptualized from the beginning rather than from existing research-validated materials. For this, data were used from investigations, both formal and informal, conducted over many years through faculty members' experiences in teaching the course, including student responses obtained from past assessments, and knowledge on the state of the fields in which they work. The tutorial was developed from the foundation provided by these data.

The guided-inquiry teaching-learning sequences in the tutorials consist of a variety of questioning and probing methods to guide students in deeper engagement and help them think critically through common difficulties. These include complex multiple-choice questions that compactly allow them to consider multiple difficulties at a time; diagrams that they are asked to analyze and annotate; passages that present correct and common alternative conceptions as explained by hypothetical students; mathematics-focused questions where appropriate and where conceptual insights can be made explicit; and prompts for students to explain their reasoning.

Under the ICAP framework, the tutorials can be used as an interactive exercise when students are allowed and encouraged to work in small groups, or as a constructive one if they work independently, as the questions and prompts in the tutorials foster the generation of new knowledge and connections as students work through them.

## **2.0 Challenges in addressing student difficulties with basics and change of basis for two-state quantum systems using a multiple-choice question sequence in online and in-person classes**

### **2.1 Introduction**

Two-state systems are often used to illustrate many rich phenomena in quantum mechanics (QM), due to their relative simplicity compared to higher dimensional Hilbert spaces. Furthermore, since we are in the midst of the second quantum revolution [28,29], they are critical to the field of quantum information science to describe the behavior of qubits, the smallest unit in which quantum information is stored and processed. Yet because of the unfamiliarity with the quantum formalism, even advanced undergraduate students can often struggle with the requisite basic concepts, e.g., the distinction between inner and outer products, translation between Dirac notation and matrix representation and the concepts of basis and changing the basis to describe a quantum state. Prior research suggests that students in quantum mechanics courses often struggle with many common difficulties. Some prior studies have focused on topical investigations, with ideas such as quantum tunneling, relativistic quantum mechanics, and the Stern-Gerlach experiments [24,108,109]. Others have investigated broader issues in introductory through graduate-level quantum education, e.g., lost opportunities for engaging motivated students, visualization of quantum states in different cases, teaching quantum on an introductory level to get students to develop intuition, or investigation of graduate student understanding of various quantum concepts [9,12,110,111]. Yet others have provided extensive overviews of student difficulties [1,2,11,46] or focused on epistemological considerations in quantum instruction [13]. Some of the other areas of student

struggle that have been investigated include the basic formalism [3,34], wavefunctions [15,37], measurement [3,7,15,42], and transferring learning from one context to other contexts [1,33]. Prior research also suggests that research-validated learning tools can effectively address such difficulties [5,16,17,112], including with concepts of basis and change of basis [113,114]. For example, Quantum Interactive Learning Tutorials (QuILTs), developed and validated by our group, have been implemented with positive learning outcomes on a number of QM topics [36,51,84]. As another example, clicker questions, first popularized by Mazur [86] using his *Peer Instruction* method [86], have similarly been shown to be effective [105]. We build on this idea by developing and validating Clicker Question Sequences (CQS), which help students learn concepts using sequences of related questions [69,71,72,106,115,73]. Here, we describe the development, validation, and implementation of a CQS intended to help students learn about the basics of two-state quantum systems, including inner and outer products, translation between Dirac notation and matrix representation, and change of basis.

## 2.2 Theoretical framework

Available class time is a pointedly limited resource in efforts to engage students in the learning process and strengthen student learning outcomes. This is particularly true for QM courses, in which the content can often be counterintuitive and appear inconsistent with students' prior experiences in everyday life as well as other classical physics courses. Instructors, therefore, must consider research-based pedagogical approaches to engage students with these challenging concepts. With this background, our theoretical framework hinges on two different aspects of

research-based pedagogical approaches, the balancing of innovation and efficiency [85] and taking advantage of peer interaction.

Schwartz et al.'s Preparation for Future Learning (PFL) framework [85] emphasizes the importance of both efficiency and innovation in instruction. The concepts of efficiency and innovation have been interpreted in different but related ways; e.g., efficiency can be used to refer to students' facility with widely applicable routine tasks that are retrieved frequently, such as mathematical manipulations that may be necessary but are not considered central to conceptual aspects of QM. On the other hand, innovation is the ability to apply existing knowledge to novel situations. Exploratory labs are examples of environments that are intended to maximize innovation. The authors observe that, in most traditional classrooms, efficiency is emphasized while innovation is typically disregarded, and suggest that balancing innovation and efficiency in learning activities can improve conceptual understanding and transfer of knowledge to new contexts [85].

In a separate publication, Schwartz and Martin used "efficiency" to describe aspects of several different instructional conditions, noting that invention tasks are significantly less efficient than more direct ways of delivering instruction, including lectures and readings, which are faster and more effective at teaching students to produce the correct answers. However, there can be a "time for telling" (or lecture) after students have struggled with innovative invention tasks, and they shed light on helpful effects that productive struggle during invention tasks can provide for students' learning [116]. Nokes-Malach and Mestre [89] analyzed Schwartz et al.'s study further using their own model of knowledge transfer and proposed that allowing students to struggle through invention tasks imparts a "hidden" effectiveness in several key ways. First, the particular framing and learning environment may prime students to operate in a more mastery-based

orientation rather than a performance-based one; a mastery orientation is one in which students are interested in deeply understanding the material, while performance orientation implies students' desire to, e.g., get a good score or pass the course [117]. Similarly, the students in the condition resembling direct instruction may use problem-solving steps via a template without comprehensively considering the reasoning for each step. The authors concluded that, in that study, as a result of their deeper, mastery-oriented and highly-engaged struggle, students in the invention condition may have identified difficulties during their problem-solving session and remained unsatisfied with their initial solutions, which may have primed them to learn better subsequently. As a result, further instruction led them to outperform direct instruction students even when both groups were subsequently given access to a common resource explaining the concepts. Meanwhile, without explicitly knowing about such possible holes in their understanding, the direct instruction students would not have been actively primed to be looking to fill in such holes [89].

Having outlined the benefits of incorporating innovation, the aforementioned fixation on efficiency in most solely lecture-based classrooms leads to what Schwartz et al. term "routine experts," who are fluent in a specific type of task but struggle with transferring their learning to new situations, rather than "adaptive experts" who are able to complete a wide variety of tasks requiring transfer of knowledge in their area of expertise with speed and accuracy. A pure focus on innovation leads to another issue: the "frustrated novice," who without guidance and scaffolding support that efficient aspects of instructional design may provide is unable to make any meaningful progress on problems in a given amount of time. Thus, the authors conclude, developing students' expertise in a domain requires balancing of efficiency and innovation axes in the instructional design, outlining an "optimal adaptability corridor" by which they can develop competencies to become "adaptive experts" with the least amount of wasted effort [85].

Integrating a CQSs with lectures and using collaborative learning and whole-class discussions when incorporating research-based CQSs offer one mechanism for balancing efficiency and innovation in instructional design. This type of instructional design, in which CQS and lectures are integrated and students are provided opportunities to engage in discussions with peers and instructors, can help students develop competencies through the efficiency of lectures paired with the innovation of productive struggle during CQS administration and discussion. This approach can provide students opportunities to have small-group or whole-class discussions about specific concepts that may reinforce the benefits of lectures and deliberations. Moreover, when student difficulties are used as resources [118] to guide the development, validation, and evaluation of the CQS, prominent patterns of student thought can be efficiently addressed by letting students innovatively explore the concepts that need strengthening. Students typically reach a consensus of the correct answer about 80% of the time, and for the remaining cases, the instructor can clarify what the correct answer is.

Collaborative learning can be productive in physics classrooms [86,87], particularly when individual accountability and positive interdependence have been suitably incentivized. Methods to incentivize students include grade incentives and appropriate framing of the activities, such as by making clear to students that, if they strive to develop their communication skills and ability to work with others while in the low-stakes environment, they are setting foundations for future personal and professional growth. Under such conditions, there are several mechanisms by which collaborative learning may be beneficial. Firstly, although students may not be at the same point in their learning of a particular concept, they may still understand one another's difficulties better than the instructor can, since they have all learned the concept recently. This makes them likely to more effectively and efficiently clarify issues related to the difficulties and solve the problem

together [90,91]. Secondly, explaining their thoughts to one another demands full engagement and analysis of their own thought processes in order to effectively communicate, so efforts to explicitly present their knowledge give clarity to the students doing the explaining, as well as those who initially articulated their difficulties during collaborative learning. The fine-grained parsing of each member's relevant knowledge may lead to more careful consideration of the problem and even yield insights that were not obvious beforehand (leading to co-construction of knowledge) [92]. Thirdly, collaborative learning often creates inherent expectations of accountability for each student, and it has been suggested that students are often more comfortable interacting with peers about their difficulties than with an authority figure [93]. Such interactions can include communicating their difficulties and clarifying their thought processes related to concepts in a domain. Moreover, through practice in collaboration, students not only can develop more comfort and competencies in their communication skills [94–98], but collaborative learning has also been found to positively affect students' self-efficacy and other motivational beliefs, which are associated with better student performance and persistence in STEM courses [99,100]. Additionally, in-class collaboration can forge connections between students that they can continue to use as resources outside of class, for example, when completing homework assignments.

Peer collaboration has been shown to be an effective method for students to learn in previous work for a variety of contexts [87,101], including in physics [102,103]. Students' performance on conceptual physics questions can receive a substantial boost from working in pairs compared to when only working individually, even if only one or neither student knew the correct answer for any given question. In particular, student pairs in which neither student initially provided a correct answer were able to reach the correct answer 30% of the time, a phenomenon known as co-construction. This effect of elevated performance even persisted when the students

were assessed again individually, pointing to significant retention [103]. There is also some evidence that these effects may apply to quantum mechanics concepts as well [83].

Emphasizing the importance of collaborative learning, Chi et al. proposed the ICAP framework, in which there are four broad modes of learning: Interactive, Constructive, Active, and Passive (ICAP). The Passive mode comprises methods of traditional and direct instruction, such as through classroom lectures, readings, or videos, in which students only engage through visual and auditory channels without any further activity. The Active mode is used in this framework to describe situations in which students are making choices or physical manipulations in their learning materials, but without generating explicit new knowledge or connections to old knowledge, which is the condition that satisfies the criteria of the Constructive mode. Finally, the Interactive mode is characterized by co-construction in small groups, which is the scheme described above, and is associated with the largest improvements in student retention and performance as compared to the Passive mode. As the wording suggests, the Interactive mode is only possible when all constituent students are already situated and working together in the Constructive mode. Broadly speaking, research shows that each successive mode subsumes the behaviors and benefits of all the modes that rank beneath it, in the descending order I/C/A/P [88,104].

The clicker questions, first popularized for use in physics courses by Mazur using the *Peer Instruction* technique, are intended to be conceptual multiple-choice questions posed to the class to which students reach a consensus by discussing in small collaborative groups. Mazur's method detailed in *Peer Instruction* falls under the Interactive mode, and has been associated with better learning outcomes including performance and retention [86,105].

The CQS questions are designed to be an Interactive and Constructive activity, requiring students to think about conceptual knowledge and their implications in new and different ways collaboratively. When students engage in discussion about the CQS questions in small groups, they work under the Interactive learning mode within the ICAP framework. Although this is the preferred mode, in the research presented here, during the online implementation, constraints imposed by time and the affordances of the technology resulted in a largely absent groupwork component for the CQS. Instead, students were simply given the questions to think about, and they answered the questions via individual polling. Therefore, under the ICAP framework, we consider the students to be in the Constructive learning mode while engaging with the CQS content in the online year, and in the Interactive mode during the in-person years.

## 2.3 Methods

### 2.3.1 Description of CQS

A description of the CQS, consistent with its learning objectives, is provided here. CQS 1.1-1.4 help students identify properties of two-state spin systems and spin-1/2 systems in particular. CQS 2.1-2.6 help students achieve fluency in translating between Dirac notation and matrix representation and calculating inner and outer products. Finally, using the knowledge about bases and products from the preceding question sequences, CQS 3.1-3.5 focus on helping students change the basis of a quantum state through several approaches. In particular, the three methods discussed to help students be able to change basis were direct substitution (e.g., using  $|\pm x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle)$  to replace a state expressed in the  $x$ -basis using the standard notation); the

viewing of inner products, e.g.,  $\langle \pm x | \chi \rangle$  as projections of the state  $|\chi\rangle$  along the basis vectors in a particular basis (in this case, the  $x$ -basis); and the use of spectral decomposition of unity. The clicker questions are provided in Appendix A.1, and the concepts on which they focus are summarized in Table 2.1.

### **2.3.2 Development and validation**

The CQS on the basics of two-state quantum systems is intended for use in upper-level undergraduate QM courses. During the development and validation process, we took inspiration from a QuILT on Dirac notation [119], which contained several key concepts applicable to both the basics of two-state systems and methods of changing basis. This enabled us to build on previous work without starting over completely, since much of the work in cognitive task analysis, from both student and expert perspectives, had already been completed. The QuILT focuses on the contexts of both wavefunctions in an infinite-dimensional vector space and two-state systems, while the CQS adapts the content fully to two-state systems, with an eye toward condensing the material and highlighting the most important information in light of the greater time constraints. Using the insights gained from the development of the QuILT, including student interviews and investigations in authentic classroom environments, we adapted the relevant learning objectives and questions while supplementing them with new ones. This iterative process involved the input of researchers and other physics faculty members, incorporating many perspectives to ensure maximal clarity and consistency in the wording and framing of the questions. The CQS is designed to provide students opportunities to think about common difficulties, struggle productively, and get immediate feedback from their peers and instructors. The CQS also deliberately includes concrete questions, which provide opportunities for students to apply their knowledge in specific

contexts, and abstract questions, which can help students generalize their understanding of the concepts and transfer their knowledge across contexts.

In this CQS, the questions are carefully sequenced to build on one another. For example, the same concept may be applied in different contexts or different concepts may be applied in similar contexts in two consecutive questions. Thus, students can compare and contrast the premise of consecutive questions to solidify their understanding of the concepts and build their knowledge structure. To facilitate the class discussion using peer instruction as mentioned earlier, we also added some slides prompting instructor-led discussion between some questions in the CQS, which can be used to review and emphasize the important concepts in the previous questions or discuss broader themes related to those questions.

After the initial development of the CQS, starting with the learning objectives adapted from the inquiry-based guided sequences in the QuILT as well as empirical data from student responses to existing individually-validated questions in previous years, we further validated the CQS by conducting individual interviews with five students in which they completed the pre-test, entire CQS, and post-test (the pre- and post-test are described in the “*Course implementation*” section below) using a think-aloud protocol. In these interviews, we asked students to think aloud while answering the questions to understand their reasoning, refraining from disturbing them so as not to disrupt their thought processes. After each question, we first asked students for clarification of the points they may not have made; we then led discussions with them on each choice as appropriate. The feedback from students helped in fine-tuning and refining the new questions, as well as ensuring that they were appropriately integrated with existing ones to construct an effective sequence of questions.

### 2.3.3 Course implementation

The data presented here are from administration of the validated CQS in a mandatory first-semester junior-/senior-level QM course at a large research university in the United States. The final version of the CQS was implemented in three consecutive years, one online and two in person, with some minor adjustments made between years to streamline the presentation of the concepts. The instructor who taught the online class also taught the second of the in-person classes discussed below, which enables us to draw comparisons between different classroom environments (online versus in-person) as well as different instructors.

During the online implementation via the conferencing software Zoom, the CQS was presented as a Zoom poll while the instructor displayed the questions via the “Share Screen” function. For the in-person implementations, the poll was replaced by a functionally similar classroom clicker system, and students were asked to think before discussing their responses with each other before answering each question via clickers. For each question, the instructor displayed the results after all students had voted, before a full class discussion of the validity of the options provided. The instructor made a significant effort to implement peer instruction using Zoom’s Breakout Rooms feature, but it was unclear whether the benefits of splitting the class into these smaller virtual groups outweighed the substantial time loss. In addition to taking much longer to separate and reconvene than would be necessary in person, the online environment itself has norms and barriers that inhibit free and spontaneous discussion among individuals. Because of these difficulties, the peer instruction feature was not considered a major factor in the online administration, but was realized for the in-person administrations.

To determine the effectiveness of the CQS, we developed and validated a pre- and post-test containing questions on topics covered in the CQS. The post-test was a slightly modified

version of the pre-test, containing changes such as use of different quantum states, but otherwise maintaining underlying conceptual similarity. The correspondence of concepts on the pre- and post-test to concepts discussed in the CQS is provided in Table 2.1. In both online and in-person classes, students completed the pre-test immediately following traditional lecture-based instruction on the topic. (Question 3 was added for the two in-person years, and did not appear in the online year.) Since all relevant material has been covered via lecture-based instruction before students engage with the CQS, the CQS constitutes a type of content review. After administration of the CQS over two to three class sessions, students completed the post-test. Two researchers graded half of the pre- and post-tests and, after discussion, converged on a rubric for which the inter-rater reliability was greater than 95%. Afterward, one researcher graded the remaining half of the assessments.

The pre- and post-test questions are reproduced in Appendix A.2. Correct answers are provided in bold. A detailed breakdown of student performance on the tested concepts is provided in the next section. In the closing sections, we compare the online implementation with both an in-person implementation with the same instructor as well as one with a different instructor. We also compare the two in-person implementations with each other to determine the generalizability of the CQS's usefulness.

**Table 2.1 Summary of the concepts that were covered in the CQS, listed along with the pre-test/post-test questions and CQS questions that address them.**

Concept	Pre-/post-test question	Corresponding CQS questions
Basic concepts related to basis and Hilbert space	1	1.1, 1.2
Operators are diagonal in certain bases which are eigenstates of the operators	6, 7	1.3, 1.4
Expression for $ \pm x\rangle$ states in terms of $ \pm z\rangle$ states	6, 7	2.1
Inner products are scalars	4	2.2, 2.6
Outer products are operators (matrices)	5	2.3, 2.5, 2.6

Translation between Dirac notation and matrix representation in the $z$ -basis	2, 3	2.2, 2.3, 2.4, 2.5, 2.6
A bra state in matrix representation is the transposed complex conjugate of its corresponding ket state	4, 5	2.4
Interpretation of expansion coefficients in a particular basis as projections of the ket state along the basis vectors	6, 7	3.1
Use of substitution (mathematical relation between different basis states) to express a state in another basis	6, 7	3.2
Use of the spectral decomposition of the identity operator to change basis	6, 7	3.3, 3.4, 3.5

## 2.4 Results and discussion

Students overall did very well on the post-test following CQS instruction. Some questions were also consistently easy for students on the pre-test, after traditional lecture instruction, but there are also questions on which students notably struggled.

**Table 2.2 Results of the online administration of the CQS via Zoom (online class). Comparison of pre- and post-test scores, along with normalized gain [120] and effect size as measured by Cohen's  $d$  [121], for students who engaged with the CQS ( $N = 29$ ). (Note: Question 3 was not asked in this year.)**

Question #	Pre-test mean	Post-test mean	Normalized gain	Effect size
1	83%	99%	0.93	1.03
2	91%	95%	0.40	0.14
4	86%	86%	--	--
5	47%	60%	0.26	0.31
6	69%	84%	0.50	0.37
7	50%	86%	0.72	0.86

**Table 2.3 Results of the first in-person administration of the CQS (in-person class 1). Comparison of pre- and post-test scores, along with normalized gain and effect size as measured by Cohen's  $d$ , for students who engaged with the CQS ( $N = 25$ ).**

Question #	Pre-test mean	Post-test mean	Normalized gain	Effect size
1	95%	97%	0.50	0.24

2	78%	88%	0.45	0.27
3	90%	96%	0.60	0.24
4	76%	94%	0.75	0.69
5	66%	94%	0.82	0.91
6	92%	98%	0.75	0.29
7	62%	98%	0.95	1.08

**Table 2.4 Results of the second in-person administration of the CQS (in-person class 2). Comparison of pre- and post-test scores, along with normalized gain and effect size as measured by Cohen’s  $d$ , for students who engaged with the CQS ( $N = 27$ ).**

Question #	Pre-test mean	Post-test mean	Normalized gain	Effect size
1	86%	99%	0.91	0.87
2	93%	96%	0.50	0.23
3	94%	100%	1.00	0.37
4	70%	81%	0.38	0.31
5	59%	85%	0.64	0.69
6	61%	87%	0.67	0.62
7	48%	78%	0.57	0.65

In light of the recommended interpretations of Cohen’s  $d$  effect sizes as small  $\sim 0.20$ , medium  $\sim 0.50$ , and large  $\sim 0.80$ , while also noting that both normalized gain and effect size are less meaningful when students perform well on the pre-test, we make the following observations.

We observe that students, in general, were quite proficient with questions 1-3 across all three years after traditional instruction, with small differences (e.g., in-person class 1 on question 2, in which the pre-test score was 78%; see Tables 2.2-2.4). Question 1 is about basics of Hilbert spaces and matrix representations, and their correspondence to physical observables. Question 2 asks students to change from Dirac notation to matrix representation, and question 3 was added after the first online year to investigate whether students were comfortable changing from matrix representation to Dirac notation. Reassuringly, students performed well on question 3 on pre-test after traditional instruction. In particular, on these questions, students scored very high on the pre-test, and they almost universally gave correct responses on the post-test, a promising sign that

advanced undergraduate students are comfortable with these representations and translating between them.

Question 4 asks students to calculate an inner product of two given states. Students did not have very much difficulty with this on either the pre-test or the post-test, and many correctly identified that the answer was a scalar. The most common difficulty was not taking the complex conjugate when transforming one of the ket states to its corresponding bra. Additionally, a small number of students gave vectors or matrices as answers instead of scalars. This was mostly limited to the pre-test and rectified on the post-test. This could indicate that these students needed to be refamiliarized with the rules of matrix multiplication, as some students who had difficulty with this question did not write the row and column vectors in the correct order when multiplying. These attempts resulted in such nonstandard matrix multiplication as, for example,  $(a \ b) \begin{pmatrix} c \\ d \end{pmatrix} =$

$$\begin{pmatrix} ac & bc \\ ad & bd \end{pmatrix} \text{ or } \begin{pmatrix} a \\ b \end{pmatrix} (c \ d) = ab + cd.$$

Question 5 asks students to calculate the outer product of two given states, which are the same states as in question 4. Students' most common mistake was providing the same answer as for question 4, a scalar rather than a matrix, for which zero credit was given. This was observed on many students' pre-test responses in all classes, and some students' post-test responses during the online class. Students in the in-person years performed better on the post-test, with some even writing their answers in Dirac notation, reinforcing that these students possessed a fluency between the Dirac notation and matrix representation. (The majority of students preferred using matrix representation, likely because the question had the given states in matrix representation, or because it offers more compact notation.) Those students who neglected to take the complex conjugate when finding the corresponding bra state did so for both questions 4 and 5. Additionally, some students found the bra state corresponding to the ket state other than the one indicated, but they

were given full credit if they otherwise performed the inner product correctly. Among incorrect responses, it was also common for students to provide the transpose of the correct matrix. On a related note, some provided the correct answer but continued to show the nonstandard method for matrix multiplication described above, indicating that they knew the result of such an operation but had not mastered the mechanics of matrix multiplication. It is possible that those students who unintentionally transposed the matrix did not have a functional understanding of the rules of matrix multiplication.

Overall, judging by the pre-test scores, inner products appear to be easier to grasp than outer products, which is reasonable partly because they learn about scalar products but not outer products in introductory courses. Many students simply repeated the inner product calculation when asked to find each one. However, particular emphasis on this point in the CQS was enough to help greatly reduce this difficulty.

For questions 6 and 7, students were asked to change basis. Question 6 provided a state in the  $x$ -basis to change to the  $z$ -basis, and question 7 went in the reverse direction, going from the  $z$ -basis to the  $x$ -basis. These questions were deliberately left open-ended so that students could use the method that they were most comfortable with, as the CQS went over several distinct approaches to changing basis. Most students chose to substitute  $|\pm x\rangle$  states with their expressions in the  $z$ -basis,  $\frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle)$ , which were given in the pre-test and post-test. In question 7, the reverse relationships  $|\pm z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle \pm |-x\rangle)$  were not explicitly provided. Since it is possible to add and subtract the  $|\pm x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle)$  equations to find the explicit relationships, direct substitution is still a viable method. Some students went through this algebra, while others correctly recognized that the relationship between the  $x$ -basis and  $z$ -basis is symmetrical, which was another valid justification. These extra steps, however, can be challenging for some students

and cause cognitive overload as they try to process all of the information, so question 7 was considered to be more difficult than question 6. For example, while a considerable number of students were unable to make meaningful progress on these questions (or left them blank) on the pre-test, that number was higher for question 7 than question 6 in each class. Fewer students used the spectral decomposition of the identity operator or explicitly described their work in terms of taking projections along the new basis, which could point to a lesser emphasis on these methods in class and on homework assignments.

The most common mistake after traditional lecture-based instruction on the pre-test was to simply divide the expansion coefficients in the starting basis by  $\sqrt{2}$ . Students with this type of difficulty did not recognize that such a state is not normalized. While the details of how students arrived at this result varied between students, in many cases, it may stem from them discarding some of the inner products  $\langle +x|+z\rangle$ ,  $\langle +x|-z\rangle$ ,  $\langle -x|+z\rangle$ , and  $\langle -x|-z\rangle$  in their attempts to obtain the final answer, resulting in an incomplete projection along the new basis. Some other students arrived at the conclusion that the expansion coefficients do not change when transforming from one basis to the other (i.e.,  $|\chi\rangle = a|+x\rangle + b|-x\rangle = a|+z\rangle + b|-z\rangle$ ), and this was most common among students for question 7 after correctly answering question 6. These notions were largely corrected on the post-test.

Additionally on the post-test, rather than finding  $a|+x\rangle + b|-x\rangle$  in the  $z$ -basis as asked, one student was observed to instead find the state  $b|+x\rangle - a|-x\rangle$ , which is orthonormal to the given state. This was an interesting response, as it demonstrated an understanding of what makes an orthonormal basis, and possibly the idea that one can arbitrarily construct an infinite number of bases, though it had little to do with transforming from the given basis to the target basis. It is likely that this student had simply misinterpreted the question.

Overall, students who engaged with the CQS demonstrated performance on the post-test equal to or higher than that on the pre-test for every concept tested, and this held true across both instructors and both modes of instruction. This shows that the CQS helped students in grasping the concepts, substantially so for the more difficult concepts. Students' common difficulties that were addressed by the CQS are summarized in Table 2.5.

#### **2.4.1 Comparisons between in-person and online instruction**

During the online year, post-test performance on question 5, involving the calculation of an outer product, did not improve appreciably compared to some of the other questions on which students struggled. It is unclear why many students, even on the post-test, simply reiterated their answers for question 4, thus providing a scalar rather than a matrix. (Students in this online year also did not improve on question 4, regarding the inner product, but it is worth noting that the pre-test score was the highest of the three years; see Tables 2.2-2.4.) This was the only year in which this remained a widespread difficulty even after CQS instruction. This is somewhat puzzling, as performance on all the other questions in the online class is comparable to that in the second in-person class, which was taught by the same instructor. On the one hand, this could have been a concept that did not become as clear to students even after the CQS, thanks to the difficulties posed by the online environment. On the other hand, their methods of changing basis were sound in questions 6 and 7, showing that a larger increase in performance is possible when using the CQS in an online setting (see Table 2.2). It is possible that the focus on the basis change content in the CQS was enough for students to have their difficulties reduced even in the online administration, while the content on distinguishing the inner product from the outer product did not get through to

the students quite as effectively. It could also be the case that the higher quality of class discussions during the in-person administrations helped students better understand outer products.

Aside from this, student scores from online instruction are seen to be only slightly worse or no worse than those from in-person instruction. Similarly, we have seen student performance in online classes in other studies to be on par in some respects [115,122]. However, students in the online class also all had their cameras and microphones off, possibly enabling them to consult resources that they were not intended to access, so there may also be some inflation in the scores, as was suggested in another study of student performance on content surveys [123].

In-person class 1 had the best performance out of all three years. The biggest improvements were observed in questions 5 and 7. In this year, the CQS on basics of two-state systems was administered after another CQS on the basics of Dirac notation, rather than serving as the very first CQS in the course as was the case in the other years. It is possible that students were more well-adjusted to learning from a CQS having already done the basics of Dirac notation CQS, compared to the online class and in-person class 2.

However, in-person class 1 did have more students than others show difficulty on question 2 on the pre-test after traditional instruction. While many students were able to correctly express certain states using matrix multiplication in the  $z$ -basis, a not insubstantial number gave responses along the lines of  $\frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle) \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . This difficulty was largely corrected on the post-test, and was not observed as a common response in other years.

## 2.4.2 Comparisons between in-person classes

On the whole, in-person class 1 did better of the two in-person classes on the pre-test. For example, in-person class 1 students, on average, struggled only on questions 5 and 7, while in-person class 2 students struggled on four questions (questions 4-7). However, on average, both in-person classes performed well on all questions on the post-test, suggesting that the CQS was effective in helping students in both classes. With regard to the differences, particularly in the pre-test scores, instructor-level and student-level fluctuations such as these are to be expected with different lecturing styles and course decisions. Moreover, both instructors taught with a spins-first approach, but they used different textbooks (McIntyre and a modified approach using Griffiths).

The question that students in in-person class 1 missed at higher rates than in either of the other classes was question 2, translating two ket states  $|+y\rangle$  (for which example responses follow) and  $| -x\rangle$  from Dirac notation to matrix representation. Aside from one blank answer, the incorrect responses provided  $2 \times 2$  matrices, e.g.,  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ , rather than column vectors, e.g.,  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ . These types of incorrect responses were observed in other years as well, in addition to other types, which most often took the form of some mathematical oversight, e.g.,  $\begin{pmatrix} 1 \\ i \\ \sqrt{2} \end{pmatrix}$  or  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 + i \\ i \end{pmatrix}$ . In addition to natural variance, instructor- and student-level differences may have played a part. However, incorrect responses were very much the exception rather than the norm in all classes (see Tables 2.2-2.4).

Despite the differences, both classes have clearly benefited from the CQS, just as students benefited in both modalities.

**Table 2.5 Student difficulties on the pre-test, successfully addressed by CQS instruction and reduced on the post-test.**

Difficulties	CQS #	Pre-/post-test #   comments
Basics of basis and Hilbert space	1.2	1   Some improvement, high pre-test scores
A bra state in matrix representation is the transposed complex conjugate of its corresponding ket state	2.4	4, 5   Some improvement online, major improvement in-person
Outer products in two-dimensional Hilbert space are $2 \times 2$ matrices, not scalars	2.3, 2.5, 2.6	5   Some improvement online, major improvement in-person
Changing from $x$ -basis to $z$ -basis (possible with substitution method, since the $x$ -basis states were provided in terms of $z$ -basis states)	3.2	6   Some improvement
Changing from $z$ -basis to $x$ -basis	3.1, 3.3, 3.4, 3.5	7   Major improvement

### 2.4.3 Retention and further learning after post-test solutions were made available

For the second in-person class, additional data were available to judge students' retention of the concepts that they had learned over the course of the CQS. Students took a midterm exam about three weeks after the post-test which included three questions that asked about concepts covered in questions 2, 6, and 7. (In this writing, corresponding questions between the post-test and the midterm exam are given the same numbers.) Questions 2 and 7 were isomorphic to those asked on the pre- and post-tests, while question 6 was in the context of quantum measurement (which the students had learned in the three weeks between the post-test and midterm exam), which nonetheless required doing a change of basis identical to question 6 on the pre- and post-test. These questions are reproduced in Appendix A.3. Table 2.6 shows that when tested on concepts from questions 2 and 6, performance has not changed appreciably, and indeed that for question 7, performance has improved nearly to full correctness.

As seen in Table 2.6, nearly all students for question 7 were able to demonstrate fluency with a method of changing basis. However, this question dealt with changing between the  $y$ - and  $z$ -bases, which do not exhibit a symmetrical relationship the way the  $x$ - and  $z$ -bases do. Since this subtlety was not emphasized in the CQS or the original pre- and post-test questions, those students who proceeded as though the relationship was symmetrical were not penalized.

Question 6 was framed as a question about quantum measurement, in which students were asked about the probability of measuring one of the outcomes in the  $z$ -basis for a state written in the  $y$ -basis. Again, as discussed above, some students treated the question as though the state was given in the  $z$ -basis and the measurement basis was the  $y$ -basis, which is the opposite direction from the one intended and would have yielded an answer of  $\frac{1}{2}$  (the correct answer was  $\frac{49}{50}$ ). Students who showed a correct procedure to calculate the incorrect answer were given the benefit of the doubt as to whether they read the given state correctly. Our rubric also avoided penalizing students for thinking that the  $|\pm y\rangle$  states are symmetrical with respect to the  $|\pm z\rangle$  states (the way the  $|\pm x\rangle$  and  $|\pm z\rangle$  states are). For example, one student explicitly wrote “This [changing basis from  $|\pm z\rangle \rightarrow |\pm y\rangle$ ] is the same as changing basis from  $|\pm y\rangle \rightarrow |\pm z\rangle$ .” Thus, students who showed their work received full credit for either answer; no students showed their work to arrive at any other answers. A few students wrote an answer of  $\frac{1}{2}$  without showing their work, which could earn them only partial credit. In these cases, it was unclear whether this answer came from the calculation or by incorrectly assuming that a measurement of any observable whose corresponding operator’s eigenbasis is not the given basis would always result in either outcome with a probability of  $\frac{1}{2}$  (e.g., a system in a state written in the  $y$ -basis upon which a measurement of the  $x$ - or  $z$ -component of spin is made).

**Table 2.6 Student performance on similar questions given on a midterm exam about three weeks after the post-test, for in-person class 2 ( $N = 27$ ).**

Question #	Post-test mean	Midterm mean	Normalized gain	Effect size
2	96%	98%	0.50	0.16
6	87%	83%	-	-
7	78%	94%	0.75	0.55

## 2.5 Summary and conclusion

We find that, after traditional lecture-based instruction, students perform well on most of the basics of two-state systems such as bra states and inner products, which is encouraging, and struggle substantially more with outer products and approaches to changing basis. We find that, for those latter concepts, a research-validated CQS is highly effective at helping students across all three years. Even though students are performing very well on content covered by questions 1-3 after lecture-based instruction, we recommend that instructors still ask the clicker questions related to these concepts. The questions are written to build on each other, so these would serve as a good warm-up to get students ready for the later questions, without taking very much time.

Moreover, we find that it was not uncommon for students to have difficulty with the rules of matrix multiplication relevant for quantum mechanics, so students' linear algebra preparation cannot necessarily be taken for granted. It was also common for students to add up the matrix elements in an outer product to get a scalar, indicating a conflation with the inner product that is rather understandable, given the visual similarities of the two in Dirac notation. In general, students navigated these concepts substantially better after engaging with the CQS. Interestingly, traditional

instruction does not appear particularly successful at teaching the outer product, as many students were confused about this concept on the pre-test.

The most difficult concepts covered by the CQS involved changing basis, which were evaluated by questions 6 and 7 on the pre- and post-tests. Common mistakes included simply dividing the state by  $\sqrt{2}$  or answering that the expansion coefficients are the same in both bases. On the post-test, however, the vast majority of students correctly expressed the given state in the desired basis.

In summary, a CQS on the basics and change of basis of two-state systems can be very helpful in reducing the prevalent student difficulties, independent of an online or in-person learning environment, or instructors' individual choices in structuring the course. In all cases, students exhibited moderate to excellent improvement on concepts that they had previously struggled with, including the nature of outer products in matrix representation and procedures of changing basis.

## **2.6 Ethical statement**

This research was carried out in accordance with the principles outlined in the University of Pittsburgh Institutional Review Board (IRB) ethical policy, the Declaration of Helsinki, and local statutory requirements. Informed consent was obtained from all interviewed students who participated in this investigation.

## 2.7 Acknowledgments

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### 3.0 Challenges in addressing student difficulties with time-development of two-state quantum systems using a multiple-choice question sequence in virtual and in-person classes

#### 3.1 Introduction

The time-evolution of a quantum state is an important concept in quantum mechanics and appears in many fields of active research, including the growing field of quantum information science. Since it draws on prerequisite knowledge of quantum states and the Hamiltonian of the system, the concept can be challenging for students to grasp on a first exposure. At the advanced undergraduate level, time-evolution of a quantum state is introduced with a time-independent Hamiltonian  $\hat{H}$ . The state as a function of time  $t$  must satisfy the time-dependent Schrödinger equation  $i\hbar \frac{d}{dt} |\chi(t)\rangle = \hat{H} |\chi(t)\rangle$ , which for a time-independent Hamiltonian is equivalent to applying the time-evolution operator  $e^{-\frac{i\hat{H}t}{\hbar}}$  to the initial state:  $|\chi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}} |\chi(t=0)\rangle$ . When applied to the corresponding energy eigenstate  $|\chi_n\rangle$ , in accordance with the time-independent Schrödinger equation  $\hat{H} |\chi_n\rangle = E_n |\chi_n\rangle$ , the operator  $\hat{H}$  yields the eigenvalue  $E_n$ . Therefore,  $e^{-\frac{i\hat{H}t}{\hbar}} |\chi_n\rangle = e^{-\frac{iE_n t}{\hbar}} |\chi_n\rangle$ . Since the energy eigenstates evolve in time via a trivial overall time-dependent phase factor, they are also known suitably as “stationary states.”

As an example, a two-state system with Hamiltonian  $\hat{H} = C\hat{S}_z$  with a dimensionally-appropriate constant  $C$  will satisfy the time-independent Schrödinger equation for two stationary states:  $\hat{H}|z\rangle = \frac{C\hbar}{2}|z\rangle$  and  $\hat{H}|-z\rangle = -\frac{C\hbar}{2}|-z\rangle$ , where  $|z\rangle$  and  $|-z\rangle$  represent the state with the  $z$ -component of its spin pointing “up” and “down,” respectively. Any initial state can be expressed in the energy eigenbasis as  $|\chi(0)\rangle = a|z\rangle + b|-z\rangle$ , with  $|a|^2 + |b|^2 = 1$ . Applying the time-

evolution operator to the two energy eigenstates replaces the Hamiltonian with the corresponding eigenvalue, so the state after time  $t$  would be  $|\chi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\chi(0)\rangle = ae^{-\frac{iEt}{\hbar}}|z\rangle + be^{\frac{iEt}{\hbar}}|-z\rangle$ . If, however, the initial state is expressed in some other basis, one can obtain the state at time  $t$  by first re-expressing the initial state as a superposition of energy eigenstates before introducing the time-dependent phase factors to each term.

To become proficient at determining a state at time  $t$  given the initial state in some basis, students must be adept at several different tasks. These include being able to recognize whether a state is an eigenstate of the Hamiltonian, converting a state to the energy eigenbasis should the initial state be given in any other basis, and correctly applying the time-evolution operator. Students also must recognize that different energy eigenstates generally correspond to different eigenvalues. The convergence of all these challenging concepts, as well as possible unfamiliarity with the meaning of the complex exponential itself, can place significant demands on students' cognitive resources. This may make other consequences of the Hamiltonian's central role in the evolution of a quantum state less obvious, e.g., the expectation value of any observable without explicit time-dependence does not depend on time in a stationary state.

Prior research suggests that students in quantum mechanics courses often struggle with many common difficulties [1,2,9–13,44,46], including with the basic formalism [3,34], notation [14], wavefunctions [15,37], the nature of probability [4], measurement [3,7,15,42], and transferring information to other contexts [1,33]. For such difficulties as those described, research-validated learning tools can effectively help students develop a robust knowledge structure [5,6,16,17,70,112]. For example, quantum interactive learning tutorials (QuILTs) have been developed, validated and implemented with encouraging results on many topics in quantum mechanics, including quantum measurement of physical observables [43,55,60], addition of

angular momentum [45], perturbation theory and corrections to the energy spectrum of the hydrogen atom [63–65,67], systems of identical particles [15,66], quantum key distribution [75], Larmor precession [47], as well as the uncertainty principle and Mach-Zehnder interferometer [36,51]. Similarly, clicker questions, first popularized by Eric Mazur using his *Peer Instruction* method, are conceptual multiple-choice questions presented to a class for students to answer anonymously, individually first and again after discussion with peers, and with immediate feedback [86]. They have proven effective and are relatively easy to incorporate into a typical course, without the need to greatly restructure classroom activity or assignments [105]. When presented in sequences of validated questions, they can systematically help students with a particular theme that they may be struggling with. Previously, such clicker question sequences (CQS) have been successfully developed, validated and implemented on several key topics in quantum mechanics [69,71–73,106]. Critically, the time-evolution of a quantum state has also been identified as a common difficulty [3,21,22,84,124], but there have not been as many materials developed for time-evolution as for some other topics. Here we describe the development and validation of a CQS intended to help students learn time-evolution of two-state quantum systems.

### **3.2 Methodology**

This research was carried out in accordance with the principles outlined in the University of Pittsburgh Institutional Review Board (IRB) ethical policy.

The CQS targets upper-level students in junior-/senior-level quantum mechanics courses. The data presented here are from implementation in a mandatory junior-/senior-level course at a large research university in the United States. To develop and validate this CQS, we studied the

learning objectives and goals of the QuILT and CQS on similar topics that had previously been developed [36,47,73]. Taking inspiration from the pre- and post-tests validated alongside those QuILTs, we made adjustments to questions to specifically address the time-development of two-state systems. Additional inspiration came from questions from other sequences, including those focused on the time-development of quantum systems in the context of Larmor precession [47].

Additionally, we took advantage of much of the cognitive task analysis based on interviews, from both the expert and student perspectives, as well as the scaffolding that had been incorporated into the aforementioned QuILT. We focused on condensing this material to ensure that the CQS can be administered in the limited class time. To that end, we prioritized basic conceptual knowledge and specific consequences that students often find difficult, provided checkpoints at which the instructor should explain some broader themes related to the previous questions, and avoided burdensome calculations.

After we conceptualized the most important features that students should know about the time-development of quantum states, as well as a suitable organization and presentation of these features, we drafted, discussed, and iterated questions many times among ourselves to minimize unintended interpretations. We standardized terminologies and sentence constructions while simplifying them as much as possible to avoid causing cognitive overload for students. We also paid specific attention to the answer choices for each question. In some instances, we chose to pivot to a different learning goal after discussion revealed a more fundamental problem. Overall, we attempted to ensure that students would understand the questions unambiguously.

We aimed to address common stumbling blocks and emphasize key features that students may have missed in the large information content of a typical lecture. The thirteen questions in the CQS focused on the following four learning goals: identifying the basic properties of the energy

eigenstates or stationary states (CQS 1.1-1.2) transforming from an initial state to its time-evolved state (CQS 2.1-2.5); expressing a state in the energy eigenbasis before applying the time-evolution operator (CQS 3.1-3.4); and calculating the time-dependence of the expectation values of various observables (CQS 4.1-4.2). We designed several questions specifically to address certain student difficulties that have previously been found. The questions used in the CQS are reproduced in Appendix B.1.

The CQS was first implemented virtually in the fall of 2020 and was followed up by an in-person implementation in the fall of 2021, for which the major additional feature was Peer Instruction. We note that the instructors were different between the two classes. The CQS was administered in the virtual setting of 2020 as a Zoom poll while the instructor displayed the questions via the “Share Screen” function. For each question, the instructor displayed the results after all students had voted, before systematically discussing the validity of the options.

In a typical in-person classroom setting, students have easy access to one another to discuss their thinking in small groups. This, however, proved less feasible in the virtual instructional setting in 2020. When a majority of students selected an option that involved alternative conceptions, the instructor would give a hint and allow the students to vote again, or ask for volunteers to explain the reasoning behind their choices to serve as the hint. These cases, however, predominantly consisted of the student volunteers or instructor speaking to the whole class. As such, the Peer Instruction method as proposed by Mazur [86] was not implemented in full in the virtual classes, while it was achieved in the in-person classes in 2021.

To determine the effectiveness of the CQS in helping students overcome these common difficulties, we developed and validated a pre- and post-test, which had questions that were either taken directly from the CQS or on topics covered in the CQS. The post-test was a slightly modified

version of the pre-test, containing changes such as a shift from eigenstates of  $\hat{S}_x$  to eigenstates of  $\hat{S}_y$ , but otherwise remaining conceptually similar. In both virtual and in-person classes, students completed the pre-test immediately following traditional lecture-based instruction on the topic. After administration of the CQS, which took place over the course of three lecture sessions, students completed the post-test. For both, they were given a 25 min period at the end of the class session. Two researchers graded the pre-test and post-test and after discussion converged on a rubric, using which the inter-rater reliability was greater than 95%. Questions Q1, Q2, Q5 and Q6 provided students three possible answers from which to choose, and credit was awarded for correctly selecting or omitting each answer, for a total of up to three points. Questions Q3 and Q4 on the pre- and post-test were scored with two points split between answer and reasoning. On these questions, students were required to write their final answer without the operator  $\hat{H}$  to receive full credit. The pre- and post-test questions are in Appendix B.2.

### **3.3 Results and discussions**

#### **3.3.1 Lessons learned from virtual CQS administration**

The CQS in its final iteration (see Appendix B.1) consisted of four subsections, each with questions focused on a specific learning goal. The two questions CQS 1.1-1.2 constituted a short primer and review that focused on identifying the basic properties of energy eigenstates, or stationary states. CQS 2.1-2.5 focused on transforming from an initial state to its time-evolved state. CQS 3.1-3.4 focused on expressing a state in the energy eigenbasis before applying the time-

evolution operator. Finally, CQS 4.1-4.2 focused on unpacking and interpreting the time-dependence of the expectation values of various observables given an initial state.

Of the student difficulties intended to be addressed by the CQS, an item-by-item analysis indicated which ones students struggled with most. The lowest correctness percentage on any question was 27%, and the highest was 79%, rounded off for a class of  $N = 29$  students. Some questions received a large majority of correct answers, but no question appeared to have an obviously correct answer selected by all students.

CQS 1.1 asked students to define a stationary state, and the results indicated that students did not fully understand the criteria for a stationary state. An eigenstate of *any* operator corresponding to a physical observable was a popular incorrect choice. Most students selected this in addition to an eigenstate of the Hamiltonian of the system, which is one of the correct choices for CQS 1.1, but not the only one. The prevalence of this difficulty is supported by prior research [22,34]. Furthermore, students initially answered that a superposition of stationary states must itself be a stationary state (CQS 1.2). That said, given a hint from the instructor noting that *any* state could be expressed as a superposition of stationary states and an opportunity to reconsider, more students selected the correct answer on the second round of polling.

CQS 1.2 asked students to select possible stationary states for a given system. Though 23% of the students had answered CQS 1.1 noting that an eigenstate of any operator is a stationary state, a higher percentage of students (32%) answered in CQS 1.2 that an eigenstate of  $\hat{S}_x$  is a stationary state for a system with Hamiltonian  $C\hat{S}_z$ . In the second round of polling, the percentage of students who incorrectly noted that the eigenstate of  $\hat{S}_x$  is a stationary state increased to 35%. It is possible that students got confused between different components of spin angular momentum, in light of previous comments made in class by the instructor that none of  $\hat{S}_x$ ,  $\hat{S}_y$ , or  $\hat{S}_z$  are inherently special.

The topic of CQS 2.2 was the time-development of a state that is not a stationary state. For this question, many students also initially incorrectly assumed that applying the unitary time-evolution operator results in a single time-dependent phase factor  $e^{-\frac{iE_{\pm}t}{\hbar}}$  to the initial state, rather than using a different phase factor for each energy eigenstate [12,34]. Encouragingly, this tendency was observed less frequently in CQS 2.3 and CQS 2.5. In CQS 2.3, which asked about the time-development of an initial state  $|y\rangle$ , the answer choices are conceptually similar to those in CQS 2.2, while CQS 2.5 gave students several possible options from which to choose the correct expression for the time-development of a generic state. While the percentage of students choosing the correct answer remained relatively constant between 60-70%, this difficulty pertaining to attaching an overall phase  $e^{-\frac{iE_{\pm}t}{\hbar}}$  to the state was selected with decreasing frequency. For CQS 2.5 specifically, the predominant incorrect answer omitted one of the correct choices rather than selecting the option consistent with this difficulty.

Many students were not comfortable with the properties of stationary states. In CQS 2.4, which had answer choices conceptually similar to those of CQS 2.2 but asked about an energy eigenstate, a sizable fraction believed that at least one of the choices was untrue, even though all of them were in fact correct statements. Some students appeared wary, after the preceding question, of multiplying the state by an overall time-dependent phase factor, even though the state in question is a stationary state. Others refrained from correctly answering that the outcome of a measurement of the observable  $S_z$  is independent of time for this initial state. This could indicate difficulty in transferring a previous concept, the fact that a system in a stationary state remains in that state for all times  $t$ , to how this concept impacts measurements performed on this state at a time  $t$ . It is also possible that some students avoided choosing one or more of these options simply because they did not recognize that  $|z\rangle$  is a stationary state in this problem. In CQS 2.5, some

students were not comfortable taking the extra step from the correct expression  $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}}(a|z\rangle + b|-z\rangle)$  to the more explicit expression  $|\chi(t)\rangle = ae^{\frac{-iE_+t}{\hbar}}|z\rangle + be^{\frac{-iE_-t}{\hbar}}|-z\rangle$ , applying the time-evolution operator to each energy eigenstate in the superposition.

Additionally, students sometimes did not recognize when it was necessary to change the basis. This difficulty was first seen in CQS 2.3, then explicitly in CQS 3.2, in which the correct approach to find the time-evolved state for the system was to convert from the given basis to the  $z$ -basis (energy eigenbasis) before applying the time-evolution operator. In both questions, some students answered the questions by applying the time-evolution operator without changing the basis. For CQS 3.2, in which the given basis was the  $x$ -basis (which was not the energy eigenbasis), 45% of students selected “all of the above.” This was the same proportion as those who selected the correct answer, which omitted a distractor choice involving the eigenstates of  $\hat{S}_x$  written in the time-evolved state instead of the eigenstates of  $\hat{S}_z$ . Whether this was a careless mistake, or a genuine unawareness that the “energy eigenstates” mentioned in option III of CQS 3.2 were not in fact the basis states used in option II, was not clear. After the instructor allowed some students to voice their thoughts and provided hints, the class showed marked improvement in the subsequent CQS 3.3-3.4, both of which have initial states that required a change of basis before applying the time-evolution operator.

CQS 4.1-4.2 asked students to select the observables with time-independent expectation values for the given system, and the two questions illustrated the deep student difficulties with regard to expectation values of different observables in a given state. Each question invoked a specific consequence of Ehrenfest’s Theorem: CQS 4.1 addressed the fact that an observable whose operator commutes with the Hamiltonian will have a time-independent expectation value, and CQS 4.2 covered the fact that the expectation values of all observables (with no explicit time-

dependence) will be time-independent in a stationary state. For CQS 4.1, in which the system was in a non-stationary state, 50% of the students answered correctly, with the remaining students being split among the remaining answer choices. For CQS 4.2, in which the system was in a stationary state, students selected primarily between only two answer choices. The instructor gave students two chances to answer this question, but the distribution of selected answers remained very similar across both chances. 54% of students selected the correct answer on the first attempt, and 57% did so on the second. Most of the incorrect responses for CQS 4.2 selected only energy and  $S_z$  while omitting  $S_x$ , whose corresponding operator does not commute with the Hamiltonian. These students had learned from the previous discussion that observables whose operators commute with the Hamiltonian have time-independent expectation values; however, they did not realize that the expectation values of all observables are time-independent because the initial state is a stationary state. Though it is possible that some students did not notice that the given state was a stationary state, this oversight may only be true for the first attempt, since the instructor pointed out before proceeding with the second round of polling that the state was indeed a stationary state.

Table 3.1 provides a summary of the student difficulties discussed in this section.

**Table 3.1 The difficulties addressed by the CQS questions, which are found in Appendix B.1.**

Difficulty	CQS #
1. Any linear combination of stationary states is also a stationary state	1.2
2. Difficulties with distinguishing between stationary states and eigenstates of <i>any</i> operator corresponding to an observable	1.1, 2.1
3. Correctly recognizing that $ \chi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}} \chi(t=0)\rangle$ , but not being able to write $ \chi(t)\rangle$ with only the eigenvalues of $\hat{H}$ in place of the operator $\hat{H}$	2.5
4. Replacing the operator $\hat{H}$ with one eigenvalue, e.g., $E_{\pm}$ , in the time-evolution operator $e^{-\frac{i\hat{H}t}{\hbar}}$ , resulting in an overall phase	2.2, 2.3
5. Not recognizing that a stationary state evolves via a trivial overall time-dependent phase factor	2.4
6. Not recognizing that a stationary state will yield the same probabilities of measurement outcomes at time $t = 0$ and after a time $t$ for all observables (with no explicit time-dependence)	2.4

7. Not focusing on the Hamiltonian and appropriately changing basis to find the state after time $t$ in terms of energies	2.3, 3.1, 3.2
8. Mistakes in the process of changing basis	3.3, 3.4
9. Time-evolution of expectation values: students unable to grasp that if $[\hat{H}, \hat{Q}] = 0$ , then $\frac{d}{dt}\langle\chi Q \chi\rangle = 0$ even if $ \chi\rangle$ is not a stationary state	4.1
10. Time-evolution of expectation values: students unable to grasp that in stationary states, none of the observables (with no explicit time-dependence) have time-dependent expectation values	4.2

### 3.3.2 Lessons learned from virtual and in-person pre- and post-test administration

In contrast with its 2020 implementation as part of virtual instruction via Zoom, the CQS was administered in person in the fall of 2021 featuring fully-realized Peer Instruction [86]. The pre-test and post-test results (see CQS questions in Appendix B.1), as well as normalized gain [120] and effect sizes [121], are listed in Tables 3.2 and 3.3 for virtual and in-person classes, respectively. We are encouraged that students in both virtual and in-person classes appear to have benefited from the CQS: across both years, from the pre-test to the post-test, effect sizes for questions ranged from 0.34 to over 1. A table that shows the concepts from the CQS on which students are evaluated in the pre- and post-test is provided in Table 3.4.

**Table 3.2 Results of the 2020 virtual administration of the CQS via Zoom. Comparison of pre- and post-test scores, along with normalized gain [120] and effect size as measured by Cohen's  $d$  [121], for students who engaged with the CQS (N = 29)**

Q#	Pre-test mean	Post-test mean	Normalized gain	Cohen's $d$
1	75%	90%	0.59	0.91
2	39%	86%	0.77	1.40
3	67%	83%	0.47	0.45
4	40%	71%	0.51	0.72
5	67%	86%	0.59	0.84
6	71%	85%	0.48	0.68

**Table 3.3 Results of the 2021 in-person administration of the CQS. Comparison of pre- and post-test scores, along with normalized gain and effect size as measured by Cohen's *d*, for students who engaged with the CQS (N = 25).**

Q#	Pre-test mean	Post-test mean	Normalized gain	Cohen's <i>d</i>
1	64%	75%	0.30	0.34
2	41%	60%	0.32	0.54
3	34%	88%	0.82	1.53
4	18%	82%	0.78	2.01
5	64%	79%	0.41	0.50
6	60%	83%	0.57	0.83

**Table 3.4 Relations between pre- and post-test questions and the questions in the CQS.**

Q#	CQS#	Comments
1	2.2	Same question for pre-test/post-test and CQS.
2	1.2	Near transfer; one answer choice is the same as in CQS, but the other two are different.
3	2.5	Near transfer; multiple-choice in CQS to student-generated response on pre-test/post-test.
4	3.1, 3.2, 3.3	Near transfer; multiple-choice in CQS to student-generated response on pre-test/post-test.
5	4.1	Near transfer; changed various observables ( $S_x \rightarrow S_y$ , etc.).
6	4.2	Near transfer; changed various observables.

The overall average post-test scores after administration of the CQS improved to 83% in the virtual implementation and 77% in the in-person implementation. The somewhat lower average post-test scores for in-person classes in which Peer Instruction was incorporated fully may be for a variety of reasons, including the fact that students were different. However, one major reason could be that the pre- and post-tests in the virtual class were not proctored. Despite being told that these were closed-book and closed-notes, students in the virtual class who had their cameras off could have consulted with those resources, unlike in the in-person class. That said, students in the

virtual class could have utilized such an advantage during both the pre-test and post-test, so the gain between the tests in both virtual and in-person classes is encouraging regardless, and many, though not all of the student difficulties addressed in the CQS were substantially reduced. Here we discuss each issue addressed, starting with cases in which student difficulties decreased significantly after CQS instruction, and then discussing one that was less successfully addressed.

### 3.3.2.1 Difficulties with stationary states in general

Questions Q1 and Q2 probed students' knowledge of stationary states, and among the highest normalized gains in 2020 were seen in these questions (see Table 3.2). It appears that students knew reasonably well by the post-test that a linear combination of stationary states is not a stationary state, as demonstrated by performance on Q2. Students were able to transform from the given initial state in Q3 to the time-evolved state with a high success rate, which shows that they were better at applying the time-evolution operator on energy eigenstates than in the pre-test. By comparison, the post-test scores on these questions were lower in 2021, but student performance did still improve significantly from the pre-test.

### 3.3.2.2 Confusion about generalized notation

During the 2020 virtual administration of the CQS, a large number of students were not able write the expression  $|\chi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\chi(t=0)\rangle$  so that it did not contain the Hamiltonian, when the initial state was given as  $|\chi(t=0)\rangle = a|z\rangle + b|-z\rangle$  with  $|a|^2 + |b|^2 = 1$  and the Hamiltonian as  $\hat{H} = C\hat{S}_z$ . However, most of the students did successfully rewrite in such a manner the expression on Q3 of the post-test, which gave a specific initial state  $|\chi(t=0)\rangle = \frac{1}{\sqrt{5}}|x\rangle + \frac{2}{\sqrt{5}}|-x\rangle$  with Hamiltonian  $\hat{H} = C\hat{S}_x$ . Work from these students shows that they followed a

procedure nearly identical to one that would be expected had the state instead been given as  $a|x\rangle + b|-x\rangle$ . The students also performed similarly well on Q3 in the 2021 in-person implementation, which is noteworthy given the much lower pre-test score, resulting in the largest observed normalized gain of 0.82 (see Table 3.3).

### **3.3.2.3 Difficulties with stationary states vs. eigenstates of any operator corresponding to observables other than energy**

Only the energy eigenstates of a system are the stationary states, but students often incorrectly associated the eigenstates of any generic operator with the stationary states. This difficulty could be exacerbated if students are shaky on the prerequisite linear algebra, without a clear grasp of what different eigenstates and operators mathematically or conceptually mean. Moreover, even if students are proficient with the linear algebra in the context of a math course, transferring that knowledge to the context of a quantum mechanics course can still be challenging. As illustrated by Q1 on the pre-test and post-test, students appeared to better distinguish between generic eigenstates and energy eigenstates after the CQS. As seen in Table 3.2 for 2020 (Table 3.3 for 2021), the correctness of this question improved from 75% (64%) to 90% (75%), with a normalized gain of 0.59 (0.30), which is at least partially due to improvement in this difficulty.

For post-test Q2, the given Hamiltonian commutes with  $\hat{S}_z$ , and the students more successfully avoided identifying the eigenstate of  $\hat{S}_x$  (written in the  $z$ -basis) as a stationary state. Additionally on this question, students also better recognized that a superposition of stationary states is not a stationary state. 39% (41%) of students answered this question correctly on the pre-test, and 86% (60%) answered correctly on the post-test, with a normalized gain of 0.77 (0.32), indicating substantial improvement for 2020 and moderate improvement for 2021, again partially

attributable to improvement in this difficulty. Similarly, students who correctly answered Q5 knew that the expectation values of only energy, and observables whose corresponding operators commute with the Hamiltonian, do not vary with time even in a non-stationary state.

#### **3.3.2.4 Replacing the operator $\hat{H}$ with one eigenvalue $E_n$**

During the virtual administration of CQS 2.2-2.5, many students selected an answer option that involved a single phase term containing energy instead of a sum of terms (distractor choice I in CQS 2.2 invokes this idea). This is at least partially due to students not being entirely comfortable with this concept, or not understanding the role of the Hamiltonian in the time-evolution of a general system. After CQS instruction, most students in both years correctly multiplied each energy eigenstate by a separate phase factor instead of a common phase factor for both Q3 and Q4. Final post-test performance was relatively strong for both questions, with normalized gains around 0.50 in 2020 and 0.80 in 2021 (see Tables 3.2 and 3.3).

#### **3.3.2.5 Difficulties with change of basis**

Tables 3.2 and 3.3 show that, once students learned the importance of working in the appropriate basis, more were able to correctly answer Q4 on the post-test compared to the pre-test. In the 2020 virtual setting, post-test performance on Q4 is a bit lower than that on Q3, at 71% and 83%, respectively. These results generally carried over to the 2021 in-person setting, with students once again scoring slightly lower on Q4 (88%) than Q3 (82%); however, in both years, effect sizes were larger for Q4 than for Q3, so their improvement on Q4 was more uniform when compared to Q3. With the exception that in Q4 students needed to express the given state in the correct basis, the two questions were identical. Based upon class discussions, the lower performance on Q4 was

mainly due to not recognizing the need for the basis change, or making a mistake in the basis change process.

### **3.3.2.6 Expectation values of stationary states**

Improvement on Q6 appears similar in 2020 and 2021 according to Tables 3.2 and 3.3; however, the number of students who selected correctly for all three choices (since Ehrenfest's theorem demands that they choose no observables' expectation values to depend on time in a stationary state) was relatively low even on the post-test. This seems to indicate a particularly resistant difficulty: students must recognize that probabilities of measurement outcomes in a stationary state do not depend on time, implying static expectation values. More scaffolding is evidently needed to help students learn this concept well. Moving forward, we would specifically include notes for instructors to encourage students to think of an expectation value as an average of a large number of measurements made on identically prepared systems, and how the probability of measuring different outcomes does not change in a stationary state. Students could also benefit from an additional discussion of Ehrenfest's theorem and the conditions that set  $\frac{d}{dt}\langle Q \rangle = 0$  for an observable  $Q$ , giving them more tools with which to process these ideas and organize their knowledge [60].

### **3.3.3 Lessons learned from virtual and in-person class discussions**

A particular advantage of the CQS is that it provides opportunity for rich class discussions that can deepen student understanding. Despite the fact that the 2020 virtual format was not especially conducive to organic discussions among students, despite efforts made to that end, the instructor still engaged the class to the best of his ability. For CQS 1.2, which addressed the

common difficulty that any superposition of stationary states is itself a stationary state, the correct answer was initially selected by only 29% of the class. Without revealing the answer, the instructor noted that the most general superposition of energy eigenstates could describe *any* possible state, and thus this option implied that every possible state is a stationary state. After this, on the second polling, the rate of correct answers rose to 48%, which is an improvement, though not a drastic one. In CQS 3.2, the Hamiltonian of the system is specified to commute with  $\hat{S}_z$ , but the initial state is given as a superposition of the eigenstates of  $\hat{S}_x$ . The instructor asked for two student volunteers to explain how the time-evolution of the state could not be expressed without the operator by remaining in that basis, and that option II, which replaced the Hamiltonian in the time-evolution operator with the energy eigenvalues without changing the initial basis, could not be correct. Finally, the topic for CQS 4.2 was the time-dependence of the expectation values of energy and components of spin in a stationary state, and despite the instructor's hint that the given state was an energy eigenstate (and thus a stationary state), the distribution of answers remained nearly identical both times the polling was opened to students. This points to the fact that this particular difficulty is robust, but it may also be attributable to the limitations of the CQS in a virtual format. While the students may not have been able to sufficiently parse the hint individually, it is likely that the performance would have improved in a typical classroom setting that would have allowed them to discuss the meaning of this hint and its consequences in small groups.

In the fall of 2021, the instructor reported lively, high-quality discussions for each clicker question. Though the post-tests do not appear to unambiguously reflect better overall student understanding as compared to 2020, the CQS was still successfully implemented as recommended by Mazur [86] and showed positive results. Students had abundant opportunities for more intimate discussions that often went for the whole time that they were allotted, and the comparison of post-

test scores between the virtual and in-person implementations do not necessarily reflect the quality of discussion that occurred during class time.

Opportunities to hold an overall class discussion about salient concepts such as these after students have voted are very important, but ensuring that instructors hold such discussions when they are recommended can be a challenge especially because time is limited. We will continue to investigate ways to encourage such discussions via checkpoints between CQS questions, even in instances when the instructor may opt not to follow our suggestions verbatim.

### **3.4 Conclusions and summary**

CQs can be effective when implemented alongside traditional classroom lectures. We developed, validated, and found encouraging results from implementation of a CQS on the topic of time-development in two-state systems in virtual and in-person classes. Post-test scores improved for every question following the administration of the CQS in both settings. It is interesting to note that, from the pre-test to the post-test, students in 2020 showed overall more improvement in the multiple-answer questions Q1, Q2, Q5 and Q6, while the 2021 students improved much more dramatically in the free-response questions Q3 and Q4; this is evident in both the normalized gains and the effect sizes (see Tables 3.2 and 3.3). As a whole, normalized gains appeared roughly uniform for the 2020 students, and noticeably larger for the open-ended questions compared to the multiple-choice ones in 2021. Meanwhile, effect sizes in 2020 were conventionally large to medium: most were over 0.70. In 2021, effect sizes were more polarized, with the multiple-choice questions giving values between 0.30-0.80, and the open-ended questions boasting remarkable effect sizes of 1.5-2.0.

Having considered these results, we have made some improvements for future administrations. One of them is the inclusion of hints that ask students if a change of basis on a particular state is required before writing the time-evolved state without the Hamiltonian operator. Another is the inclusion of specific numbers as the expansion coefficients for states written in a particular basis, in addition to the general  $a$  and  $b$  with  $|a|^2 + |b|^2 = 1$  that exclusively appeared in this CQS. Additionally, we have expanded the discussion on expectation values to address the physical interpretation of an expectation value as well as Ehrenfest's Theorem.

Regarding the somewhat surprising result that students performed equivalently or even better during virtual instruction, with inadequate Peer Instruction opportunities, we acknowledge that students may have been able to consult resources that they were not intended to during the virtually-administered pre- and post-tests. Even though they were told that it was a closed-book, closed-notes quiz, there was no way to adequately determine or enforce which resources students used when they had their cameras off. Even so, the students would have had access to those same resources during the pre-test, given after traditional lecture-based instruction, as well as the post-test, so the fact that post-test scores in the virtual class are significantly better is very encouraging. Moreover, we point out that students may have been stressed about the transition from virtual courses back to in-person learning, which may have affected performance for in-person class. Discussions with other instructors suggest that a greater number of students appear to be performing at a lower level in other physics courses after transition back to an in-person setting. In summary, the administration of the CQS in 2020 in a virtual learning context may have affected student performance compared to the in-person 2021 administration, but both classes benefited from the CQS. We note also that the results presented refer to the short- and mid-term retention of

material, while analysis of long-term retention would require future studies and a longer lapse of time.

### **3.5 Acknowledgements**

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## **4.0 Challenges in addressing student difficulties with quantum measurement of two-state quantum systems using a multiple-choice question sequence in online and in-person classes**

### **4.1 Introduction**

Quantum measurement is a foundational concept in quantum mechanics (QM) which students must learn, with applications in future studies in the field and for many related fields, including the emerging field of quantum information science [27–29]. Yet because the formalism is different from that for classical measurement and difficult to ground in everyday experience, there are many concepts related to quantum measurement that can be challenging for many students to grasp [42].

Prior research suggests that students in quantum mechanics courses often struggle with many common difficulties, and research-validated learning tools can effectively help students develop a functional understanding and build a robust knowledge structure [2–12,1,13–26]. For example, our group has developed, validated and implemented Quantum Interactive Learning Tutorials (QuILTs) with encouraging results on many topics in quantum mechanics [36,42,43]. Other commonly used research-based learning tools in physics include clicker questions [86], which are conceptual multiple-choice questions presented to a class for students to answer anonymously, typically individually first and again after discussion with peers, and with immediate feedback.

While these questions can be successfully integrated and implemented without additional technological tools, the research presented here used an electronic response system, generally referred to as “clickers,” which automatically tracked student responses in real time. When

presented in sequences of validated questions, clicker questions strive to systematically help students with particular concepts that they may be struggling with. Previously, such multiple-choice question sequences, or Clicker Question Sequences (CQS), related to several key QM concepts have been developed, validated and implemented [69,71,125] with encouraging results. As they are effective and are relatively easy to incorporate into a typical QM course, without the need to greatly restructure other classroom activities including lectures or assignments, CQSs are a promising way to help students learn challenging concepts as a supplement to traditional lectures and homework assignments.

## **4.2 Theoretical framework**

In QM courses, whose content can be difficult for students, instructors must consider research-based pedagogical approaches to engage students and help them learn these challenging, foundational concepts. Our theoretical framework for developing, validating and evaluating CQSs hinges on two different aspects of research-based pedagogical approaches, the balancing of innovation and efficiency [85] and taking advantage of peer interaction. The CQSs use research on student difficulties as resources [118] and efficiency is embedded in the way that concepts are presequenced. The CQSs also focus on providing students opportunities for innovation by means of productive struggle with new ideas through peer co-construction. While in our research presented here the concepts are first introduced through lecture-based instruction before administering the CQS, it could also be possible to structure the CQS around a just-in-time teaching scheme [53,126].

Collaborative learning can be productive in physics classrooms [86,87], particularly when individual accountability and positive interdependence have been suitably incentivized, such as through grade incentives. Peer collaboration has been shown to be an effective method for students to learn in previous work for a variety of contexts [87,101], including in physics [102]. Students' performance on conceptual physics questions can receive a substantial boost from working in pairs compared to when only working individually. In a phenomenon known as co-construction, prior research shows that student pairs in which neither student initially answered the questions correctly were able to converge on the correct answers 30% of the time, an effect that persisted when the students were assessed again individually, pointing to retention [103]. In QM courses, this rate of co-construction was about 25% [83].

Emphasizing the importance of collaborative learning, Chi et al. proposed the ICAP framework, in which there are four broad modes of learning: Interactive, constructive, active, and passive (ICAP). Only the constructive and interactive modes are concerned with the high engagement for which the CQS is designed. The main difference between the constructive and interactive modes is that, instead of providing support to students for constructing knowledge individually, the interactive mode is characterized by co-construction in small groups (realized by collaborative learning) and is shown to be associated with larger improvements [88,104].

The clicker questions, first popularized for use in physics courses by Mazur using the *Peer Instruction* technique, are intended to be conceptual multiple-choice questions posed to the class to which students reach a consensus by discussing in small collaborative groups. Mazur's method detailed in *Peer Instruction* has been associated with better learning outcomes including performance and retention [86,105], and when students engage in discussion about the CQS questions in small groups, they work under the interactive learning mode within the ICAP

framework. Although this is the preferred mode, in the research presented here, constraints imposed by time and the affordances of the technology during the online implementation resulted in a largely absent groupwork component for the CQS. Instead, students were simply given the questions to think about, and they answered the questions via individual polling. Therefore, under the ICAP framework, we consider the students to be in the constructive learning mode while engaging with the CQS content in the online year, and in the interactive mode during the in-person years.

## **4.3 Methods**

### **4.3.1 Development and validation**

The CQS on quantum measurement discussed here is intended for use in upper-level undergraduate QM courses. Here we summarize the development and validation of the CQS. We took inspiration from some of the previously-validated learning tools on quantum measurement to first determine the learning objectives. In particular, much research involving cognitive task analysis, from both student and expert perspectives, has already been conducted in the development and validation of a QuILT and a CQS on quantum measurement in the context of wavefunctions in an infinite-dimensional vector space [42,43]. A number of individual student interviews as well as investigations in authentic classroom environments have previously been conducted to develop, validate and evaluate classroom effectiveness of this QuILT. Using the insights from the QuILT to identify students' prior knowledge, their difficulties, and the scaffolding supports needed to help reduce those difficulties, we adapted the relevant learning

objectives and questions from the QuILT while also drafting and iterating new ones for measurement related to two-state quantum systems for the CQS. This process involved the input of researchers and other faculty members, incorporating many perspectives to ensure maximal clarity and consistency in the wording and framing of the questions.

The CQS is designed to help students improve their conceptual understanding of quantum measurement, so we avoided complicated calculations to reduce cognitive overload for students. Since the CQS is designed to be used in class with peer instruction, we ensured that it was of a length suitable for administration during limited class time while still covering the common difficulties students have in understanding quantum measurement. We carefully designed each alternative choice in each multiple-choice question and incorporated the common incorrect responses that we found in previous interviews and students' written responses to the QuILT and its corresponding pretest and post-tests. Thus, the CQS provides students opportunities to think about common difficulties, struggle productively, and get immediate feedback from their peers and instructors. Another feature of the quantum measurement CQS is its inclusion of both concrete and abstract questions. Concrete questions provide opportunities for students to apply their knowledge in concrete contexts, which help them learn applications of the quantum measurement concepts in specific contexts. Concrete questions are usually followed by or integrated with abstract questions, which can help students generalize their understanding of the concepts and transfer their knowledge across contexts.

In the quantum measurement CQS, the questions are carefully sequenced to build on each other. For example, the same concept may be applied in different contexts or different concepts may be applied in similar contexts in two consecutive questions. Thus, students can compare and contrast the premise of consecutive questions to solidify their understanding of the concepts and

build their knowledge structure. To facilitate class discussions after peer interaction, we developed some discussion slides between the CQS questions, which can be used by instructors to review and emphasize the important concepts in the previous questions during general class discussions on some broader themes related to those questions. These discussion slides were iterated not only amongst the researchers but also with other QM instructors.

To determine the effectiveness of the CQS, we developed and validated a pretest and post-test containing questions on topics covered in the CQS, simultaneously with and using the same process as the CQS. The post-test was a slightly modified version of the pretest, containing changes such as use of different quantum states, but otherwise maintaining underlying conceptual similarity. There are surveys that have been developed previously that ask similar questions to the ones found in the CQS, pretest, and post-test [44,127,70]. The pretest and post-test questions are reproduced in Appendix C.1.

After the initial development of the quantum measurement CQS and the pretest and post-test, starting with the learning objectives adapted from the inquiry-based guided sequences in the QuILT [29] as well as empirical data from student responses to existing individually-validated questions in previous years, we further validated them by conducting individual interviews with four students in which they completed the pretest, entire CQS, and post-test using a think-aloud protocol. In these interviews, we asked students to think aloud while answering the questions to understand their reasoning, refraining from disturbing them so as not to disrupt their thought processes. After each question, we first asked students for clarification of the points they may not have made, then we led discussions with them on each choice as appropriate. The feedback from students helped in fine-tuning and refining the new questions, as well as ensuring that they were appropriately integrated with existing ones to construct an effective sequence of questions. In the

interviews, we found that students showed some common difficulties with quantum measurement consistent with results from prior studies involving the QuILT [42,43], and also that after working through the whole CQS, their difficulties with many concepts related to quantum measurement were reduced significantly. The interviewed students also reported that they found the scaffolding provided by the sequenced questions and discussion slides helpful; these slides are for instructors to discuss various issues during class discussions after students have answered a set of CQS questions.

#### **4.3.2 Learning objectives**

The learning objectives of the CQS were inspired by those for the QuILT and a CQS on quantum measurement in the context of wavefunctions [43]. CQS 1.1-1.3 help students make the distinction between quantum measurement and the action of an operator corresponding to an observable acting on a quantum state. These questions also focus on helping students be able to describe key ideas about the result of a measurement (i.e., an eigenvalue of the operator corresponding to the observable being measured) and the state of the system after a measurement (i.e., wavefunction collapse). CQS 2.1-2.3 help students calculate the probabilities of measuring certain outcomes in a given quantum state, which may or may not be written in the measurement basis. CQS 3.1-3.2 are an extension and synthesis of these ideas, in which students have to describe the results of consecutive measurements made in different bases. The last questions in the sequence, CQS 4.1-4.2, ask students to connect quantum measurement with the concept of an expectation value of an observable and its calculation (only administered to one of the classes). The clicker questions are provided in Appendix C.2 (correct answers are provided).

### 4.3.3 Course implementation and instructor details

The data presented here are from administration of the validated CQS in a mandatory first-semester junior-/senior-level QM course, taught once per year during the fall semester, at a large research university in the United States. The final version of the CQS was implemented in three consecutive years, one online and two in person, with very minor adjustments made between years. One of the two instructors who taught an in-person class was also the instructor for the online class, which enables us to draw comparisons between different classroom environments as well as different instructors.

One instructor used McIntyre's textbook *Quantum Mechanics: A Paradigms Approach*, while the other used Griffiths's *Introduction to Quantum Mechanics*. The instructor who used Griffiths's textbook started with the chapter on formalism (chapter 3), then covered spin-1/2 in chapter 4 before going back to chapters 1 and 2. Both instructors thus covered the material pertaining to two-state systems early in the course, before introducing wavefunctions and infinite-dimensional Hilbert spaces. The two instructors were very supportive of physics education research and have implemented physics education research-based pedagogies in their classes many times.

During the online implementation, the CQS was presented as a Zoom poll while the instructor displayed the questions via the "Share Screen" function. For the in-person implementations, the poll was replaced by a functionally similar classroom clicker system and students were asked to discuss their responses with each other before answering each question. For each question, the instructor displayed the results after all students had voted, before a full class discussion of the validity of the options provided. In all implementations, students were incentivized with 80% for participation and 20% for correctness on each question. Because of the

difficulties in adapting to the online environment in a way that remained conducive to small-group student discussion, the Peer Instruction feature was largely forgone in the online administration in favor of mostly instructor-led discussions. Peer Instruction was realized for the in-person administrations. Therefore, referring to the ICAP framework, we consider the CQS a constructive activity in the online class and an interactive one during the in-person classes. The online class was taught after the emergency transition to remote courses, when instructors and students had some time to adapt and prepare for the norms and expectations of the remote environment, though outstanding norms and circumstances would likely have had some impact not present in more “normal” online teaching environments. That said, students in the online QM course did not reveal to the course instructor any additional challenges in pursuing their studies remotely, suggesting that they were still able to participate in the course reasonably well.

In both online and in-person classes, students first learned from traditional lecture-based instruction all the concepts covered in the CQS before completing the pretest. After administration of the CQS over two to three class sessions, students completed the post-test. During this time period, the instructors did not cover the material again, except during the class discussions afforded by the CQS, but students were given traditional homework sets from the course textbooks. However, in our prior research, student post-test performance is significantly improved when research-based tools are used in instruction, as compared to control groups of students who only have traditional instruction and homework [43,106]. That said, this study is quasi-experimental [128] in design, in light of these and other factors over which we did not have complete control.

As an additional point, the CQS as designed is not intended to serve as students’ first instructional experience in the quantum measurement concepts covered. In our implementations, students received lecture-based instruction to give them the preparation to engage on a deep level

with the questions. However, it is not imperative that students are prepared by lecture-based instruction. As an example, just-in-time teaching could also accomplish this [53], in which students are assigned videos or readings and given some assessment tasks based upon them before class, though we did not investigate this possibility in the present study. Thus, efficiency is embedded in this combination of lecture and CQS instruction as the CQS questions build on each other. At the same time, instructors foster innovation by providing students opportunities to productively struggle with challenging concepts via peer interaction and full class discussions. The student-to-student interactions may also allow students opportunities for co-construction of knowledge.

Students knew in advance that they would take the pretest after traditional lecture-based instruction on relevant concepts (but before the CQS) and the post-test after the CQS. Both pretests and post-tests were framed as quizzes, but students were graded on the pretest for completion and post-test for correctness. The in-class rubric for the post-test grading was more lenient than the one used for this research, so the assessments were not particularly stressful. The quizzes were closed-book and closed-notes. All classes were given roughly half a class period (about 25 min) to take the quizzes synchronously, but all students in the online group exercised the option to keep their cameras turned off. Most of the questions on the pretest and post-test were graded on an all-or-nothing scheme. We would have liked to give students opportunities to provide their reasoning, but due to concern over time constraints in administering the pretests and post-tests in the class, in addition to our work being qualitatively supported by student reasoning from numerous prior studies [1,34,42,44,46,60,70], our questions took the form presented in Appendix C.1. Question 3 was graded on a three-tiered scale (zero, half, and full credit), for which two researchers graded half of the pretest and post-tests and, after discussion, converged on a rubric for which the inter-rater reliability was greater than 95%. Afterward, one researcher graded the remaining half of the

tests. A detailed breakdown of student performance on the tested concepts is provided in the next section. In the closing sections, we compare the online implementation with both an in-person implementation with the same instructor as well as one with a different instructor. We also compare the two in-person implementations with each other to determine the generalizability of the CQS's usefulness.

#### 4.4 Results and Discussion

In this section, we present results of implementation of the final version of the CQS in three consecutive years. The concepts that the questions dealt with are summarized in Table 4.1. Results for each question, as well as normalized gains [120] and effect sizes [121], are listed by class in Table 4.2, with modality and sample size specified for each case. These data are represented visually in bar charts in Figure 4.1.

**Table 4.1 Summary of the concepts that were covered in the CQS, listed along with the pretest/post-test questions and CQS questions that address them.**

Concept	Pre-/post-test question	Corresponding CQS questions
$\hat{S}_z \chi\rangle = \frac{\hbar}{2} +z\rangle$ or $\hat{S}_z \chi\rangle = \frac{\hbar}{2} -z\rangle$ represents measurement of z-component of spin, where $ \chi\rangle$ is the state before the measurement and $ \pm z\rangle$ is the state after the measurement [incorrect statement]	1	1.1
Same as question 1, but with general Hermitian operator $\hat{Q}$	2a	1.2, 1.3
Normalization of a quantum state	2b	1.3, 2.1
Measurement in a quantum state that is not given in the measurement basis	2c	2.2, 3.2
Possible outcomes, and probabilities of measuring those outcomes, for a measurement of an observable in a given quantum state	3	1.3, 2.1

Normalized state after measurement collapse	4a, 4b	2.3, 3.1, 3.2
Successive measurements of $S_x \rightarrow S_z$	4c	3.1, 3.2
Successive measurements of $S_x \rightarrow S_z \rightarrow S_x$	4d	3.2
Expectation values of various observables in various states	5	4.1, 4.2

**Table 4.2 Results of the online and in-person administrations of the CQS. Comparison of pretest and post-test scores, along with normalized gains [120] and effect sizes as measured by Cohen’s *d* [121], for students who engaged with the CQS. For the online class,  $N = 30$ ; for in-person class 1,  $N = 23$ ; and for in-person class 2,  $N = 28$  (note: an additional question 5 was added for in-person class 2).**

Online class	Question #	Pretest	Post-test	Normalized gain	Effect size
	1	27%	63%	0.50	0.78
	2a	33%	63%	0.45	0.62
	2b	97%	97%	-	-
	2c	67%	83%	0.50	0.39
	3	92%	97%	0.60	0.24
	4a	73%	87%	0.50	0.33
	4b	80%	97%	0.83	0.53
	4c	77%	93%	0.71	0.47
	4d	67%	93%	0.80	0.70
In-person class 1	Question #	Pretest	Post-test	Normalized gain	Effect size
	1	26%	65%	0.53	0.84
	2a	26%	61%	0.47	0.73
	2b	100%	96%	-	-
	2c	87%	91%	0.33	0.14
	3	83%	96%	0.75	0.58
	4a	52%	78%	0.55	0.56
	4b	65%	91%	0.75	0.65
	4c	70%	70%	-	-
	4d	43%	57%	0.23	0.26
In-person class 2	Question #	Pretest	Post-test	Normalized gain	Effect size
	1	36%	46%	0.17	0.22
	2a	21%	68%	0.59	1.04
	2b	100%	96%	-	-
	2c	79%	86%	0.33	0.18
	3	86%	89%	0.25	0.14
	4a	54%	71%	0.38	0.37
	4b	68%	75%	0.22	0.16
	4c	46%	68%	0.40	0.44
	4d	46%	50%	0.07	0.07
	5	30%	73%	0.62	1.00

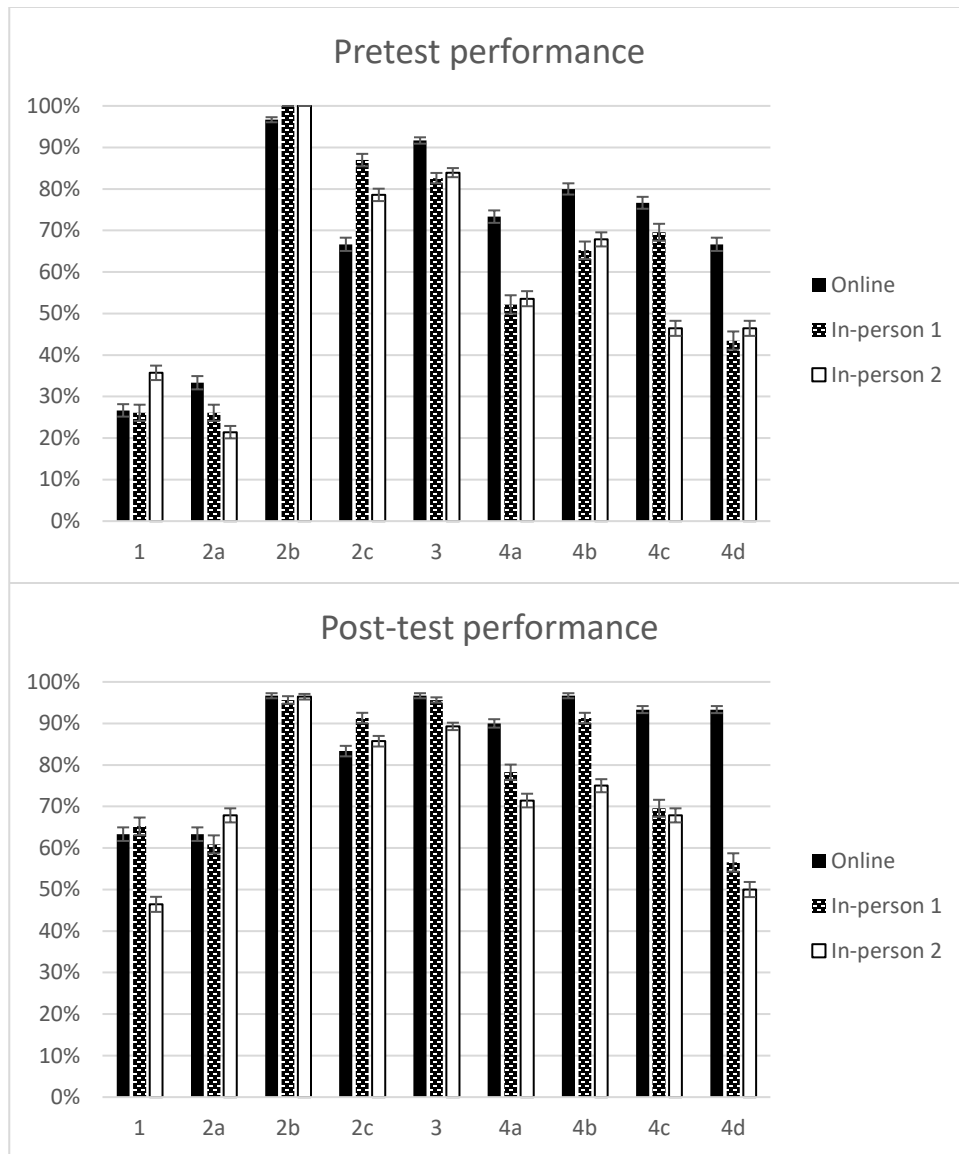


Figure 4.1 Bar charts representing the pretest and post-test scores for all classes, as reported in Table 4.2.

Error bars represent standard error.

Overall, the results are encouraging, and students performed well on the post-test, with reasonably high normalized gains. Effect sizes were generally medium to large for the online class and in-person class 1, and overall they were more modest for in-person class 2.

These results in Table 4.2 suggest that the concepts assessed by questions 2b and 3 may be understood well after traditional lecture, while the concept assessed by 1 and 2a is much more difficult for students. Although improvement is seen on student performance on questions 1 and

2a after the CQS (see Table 4.2), the underlying concept still appears to be difficult. Later we will discuss that, after struggling on both the pretest and post-test, students improved further on corresponding later midterm exam questions, after being provided solutions to the post-test questions (see subsection titled “*Retention and further learning after post-test solutions were made available*”). For questions 4a-d, the improvement is also noticeable in all classes for a great majority of the questions.

Given that the two in-person years were taught by different instructors, and that there are improvements in student performance for both years, it is clear that the CQS is beneficial for students despite the differences between instructors’ approaches. Furthermore, the CQS is beneficial in both online and in-person environments. As this study was quasi-experimental, it is possible that there are other effects in addition to the CQS itself that led to these benefits. However, in a prior study, a control group of students, who were given the post-test on quantum mechanics concepts after traditional lecture-based instruction and associated homework, were significantly outperformed ( $p < 0.0001$ ) by three sets of experimental groups of students who engaged with clicker questions after their traditional lectures and had homework similar to the control group [106]. Moreover, in another study [43], a control group of students who only had traditional lecture and homework were outperformed by students who used research-based tools after traditional lecture and homework. Thus, we believe that the CQS plays an important role in these improvements, and that homework alone does not provide the benefits seen here. Furthermore, a prior study in introductory mechanics comparing lecture vs microcomputer-based labs suggests that a research-based tool is a significant contributor to better performance, and that simply repeating or expanding upon previously covered material, such as through traditional problem-solving exercises, does not necessarily yield the same positive results [129].

In the following sections and Table 4.3, we categorize student difficulties observed during the administration of the CQS, and the extent to which they were successfully addressed.

#### **4.4.1 Action of an operator corresponding to an observable on a quantum state being confused with a measurement**

Broadly speaking, when given an observable  $Q$  with corresponding Hermitian operator  $\hat{Q}$ , eigenvalues  $q_1$  and  $q_2$  and eigenstates  $|q_1\rangle$  and  $|q_2\rangle$ , many students state that a measurement of observable  $Q$  is represented mathematically by  $\hat{Q}|\chi\rangle = q_1|q_1\rangle$  or  $\hat{Q}|\chi\rangle = q_2|q_2\rangle$  [42]. The claim is that the state before the measurement is  $|\chi\rangle$ , and after the measurement it is  $|q_1\rangle$  or  $|q_2\rangle$  depending on the result of the measurement, despite the fact that neither equation is mathematically valid. In all three implementations, students tended to gravitate strongly to this idea on the pretest (questions 1 and 2a) after lecture-based instruction and the first CQS question in this sequence (CQS 1.1). The performance on CQS question 1.1 gave the instructor an opportunity to discuss with students and deconstruct both the correct and incorrect ideas involved, such that students were able to answer correctly at a much higher rate for the following questions CQS 1.2-1.3. This improvement was reflected on the post-test, but a significant fraction of students (around 40%) continued to mistakenly rate a statement such as  $\hat{Q}|\chi\rangle = q_1|q_1\rangle$  or  $\hat{Q}|\chi\rangle = q_2|q_2\rangle$  as true (see Table 4.2). It appears that this difficulty is so persistent that students underperform on this question compared to the rest of the post-test. However, as we will discuss later, the improvement is encouraging on another assessment given two weeks after the post-test (discussed in the subsection “*Retention and further learning after post-test solutions were made available*”).

Prior studies [42,44,34,127] have shown that when the operator in question is the Hamiltonian, students struggle with this concept. In a multi-institutional study, only 22% of undergraduate and 26% of graduate students managed to correctly answer a question on the Quantum Mechanics Formalism and Postulate Survey (QMFPS) targeting this concept [70]. It is possible that students are improperly overgeneralizing the time-independent Schrödinger equation (TISE) in some ways, rather than restricting it to only the cases where the eigenvalue equation holds. We also find analogous patterns in student responses when asked about a generic operator. This may also indicate an incomplete knowledge of linear algebra, with students not knowing that an operator is a linear transformation acting on the ket state, and so dropping one of the eigenstates from the right-hand side would not serve as a prominent red flag. In fact, in interviews in which students agreed with  $\hat{Q}|\chi\rangle = q_1|q_1\rangle$  or  $\hat{Q}|\chi\rangle = q_2|q_2\rangle$ , when it was pointed out that this type of equality violates the rules of linear algebra, only then did some notice the issue, while others suggested that quantum mechanics itself might simply not follow linear algebra [1]. These students thought that quantum measurement must be represented by some equation that resembles the TISE. The overgeneralization takes this particular form likely because of the strong emphasis that a measurement of an observable in any state can only yield the eigenvalues of the operator corresponding to the observable, and that the state thereafter has collapsed to the eigenstate associated with the eigenvalue that was obtained. Coincidentally, the TISE contains entities that represent the operator, the eigenvalues, and the eigenstates of the operator, even though the TISE (and equations involving operators in general) have nothing to do with the measurement process. Ultimately, there are many surface features that tie the two ideas together, and significant care and persistence, e.g., via clicker questions and other research-based tools, is needed to disentangle them from each other.

#### 4.4.2 Normalization of a quantum state

Pretest and post-test question 2b concerned the normalization of expansion coefficients of a quantum state written in a particular basis, and students did very well on both the pretest and post-test (see Table 4.2). The normalization of a quantum state after state collapse following a measurement, which proved to be a challenging concept, was assessed by question 4a. The question started with a given state and a measurement of  $S_x$ , yielding a particular eigenvalue (e.g.,  $-\frac{\hbar}{2}$ , which indicates a collapse into the state  $| -x \rangle$ ). Some students provided the original state as their state after measurement rather than the proper eigenstate of  $\hat{S}_x$ . Other incorrect answers included keeping the expansion coefficient of this eigenstate without normalizing (e.g.,  $\sqrt{\frac{3}{10}} | -x \rangle$ ) or by attaching the eigenvalue itself to the state as the result of the measurement (i.e.,  $-\frac{\hbar}{2} | -x \rangle$ ), a response consistent with the difficulty discussed in the preceding section. Such answers were overall observed less frequently on the post-test in all three classes, indicating the effectiveness of the CQS in helping students with this concept.

Normalization of a quantum state is not necessarily an intuitive thing that first-time learners check for at every step of a calculation, so keeping coefficients like  $\sqrt{\frac{3}{10}}$  does not seem unnatural [46,127]. Indeed, in a multi-institutional study, when asked on the QMFPS [70], 17% of undergraduate students and 23% of graduate students selected such a response. Other responses like  $\pm \frac{\hbar}{2} | \pm x \rangle$ , or attachment of the appropriate eigenvalue of any quantum state, are likely closely related to the difficulty described above conflating the TISE, or any equation involving any operator that corresponds to an observable, with the process of quantum measurement.

#### 4.4.3 Measurements made in a basis different from the one given

Question 2c states that a measurement of  $S_y$  in a state  $a|+z\rangle + b|-z\rangle$  would yield an outcome of  $\frac{\hbar}{2}$  with probability  $|a|^2$ . Students were asked whether they agreed with this statement, which is false because it is necessary to change to the appropriate measurement basis before interpreting the meaning of the expansion coefficients. In the problem statement, specific attention was drawn to the fact that the measurement was not of the observable  $S_z$ , and students then should know that a basis change is necessary. Students in the in-person years did reasonably well on the pretest and also improved somewhat on the post-test, approaching full scores, i.e., everyone having relevant knowledge based upon their performance (see Table 4.2). In a multi-institutional study, 39% of undergraduate students and 28% of graduate students did not first change the basis on a QMFPS question of this type, and this appeared to be reflected in the online class, though the necessity of a basis change was not made as salient in the question in the QMFPS [70] as in our pretest and post-test questions.

In the case of measuring components of spin for two-state systems in two-dimensional Hilbert spaces, the eigenvalues are the same regardless of which component of spin is measured. This can make things less cognitively demanding, but also may pose some difficulties when generalizing to infinite-dimensional Hilbert spaces, such as with the observables position and momentum. Transforming from one basis to another, for instance, is more mathematically complex for wavefunctions in infinite-dimensional Hilbert space even though the underlying concept is the same, and students have been found to struggle with successive measurements of energy and position for wavefunctions [46], which requires changing the basis.

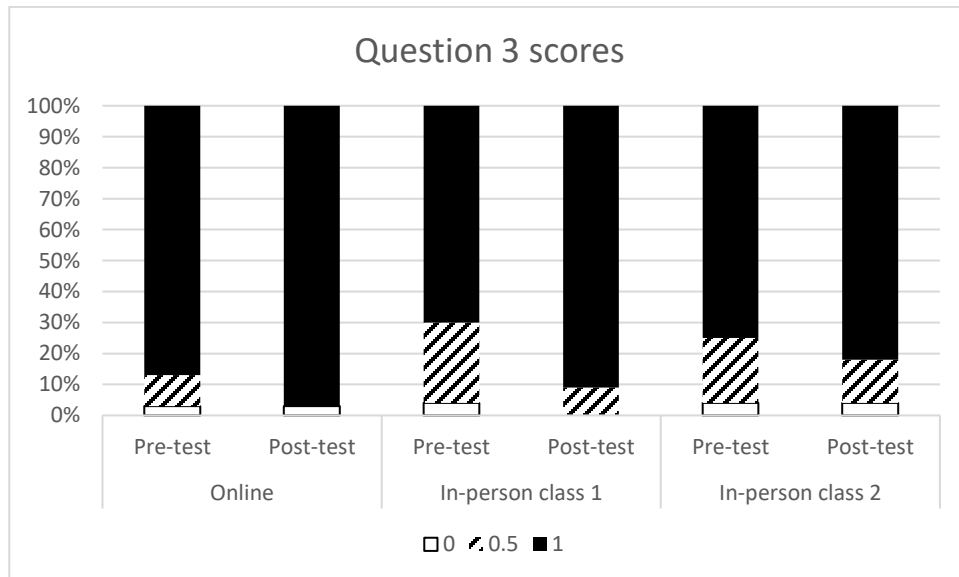
#### **4.4.4 Outcome and probabilities of a measurement outcome**

In Question 3 on the pretest and post-test, students were provided a quantum state and asked about the possible outcomes of measurement, and the probabilities of obtaining those outcomes. As with question 2b, this was also a relatively easy question for students even before engaging with the CQS (see Table 4.2). The concepts covered by these two questions are related in that normalization of a quantum state directly makes use of the fact that all probabilities must sum to 1, indicating that students had a good understanding of how to interpret probabilities of measurement outcomes in a normalized quantum state after lecture instruction. They still exhibited some improvement after CQS administration.

Students were given half credit for providing the possible outcomes, and half credit for providing the respective probabilities. The distributions of scores on this question are shown in Figure 4.2.

**Table 4.3 A table showing the shifts in distributions for question 3 from the pretest to the post-test for each class.**

Question 3 scores		0	0.5	1
Online	Pretest	3%	10%	87%
	Post-test	3%	0%	97%
In-person class 1	Pretest	4%	26%	70%
	Post-test	0%	9%	91%
In-person class 2	Pretest	4%	21%	75%
	Post-test	4%	14%	82%



**Figure 4.2 A bar chart showing the shifts in distributions for question 3 from the pretest to the post-test for each class.**

#### 4.4.5 Results of consecutive measurements of spin components

Questions 4b-4d asked students about the results of consecutive measurements of different components of spin in immediate succession. Pretest performance is seen to be quite high in the online administration, and less so in the in-person administrations (see Table 4.2). Different questions had different normalized gains and effect sizes over the three years, with some questions showing little improvement in one year but large improvements in the remaining two.

Question 4b stipulated that a measurement of  $S_x$  in a given state resulted in a particular eigenvalue, and asked students about the probability of obtaining the other eigenvalue from a measurement in immediate succession. Most students answered this question correctly on the post-test, although in-person class 2 had a higher error rate than the others. However, there was no discernible underlying pattern to students' incorrect answers.

Question 4c asked students about the results of a measurement of  $S_z$  in the given state after the preceding measurement of  $S_x$  (from question 4b) had been made. Most commonly, students who struggled changed the *original* state to the  $z$ -basis for their answers, not realizing that the state had collapsed to an eigenstate of  $\hat{S}_x$  from the preceding measurement. This is a rather complicated question that requires students to utilize several quantum measurement concepts at each step, and it is likely that they did not recognize some parts of the question that implied the measurement collapse of a state, even if they are able to recognize the state collapse when asked directly about it. Where students were observed to improve on the post-test, they did so by rather large margins, though no improvement was observed in the first in-person class.

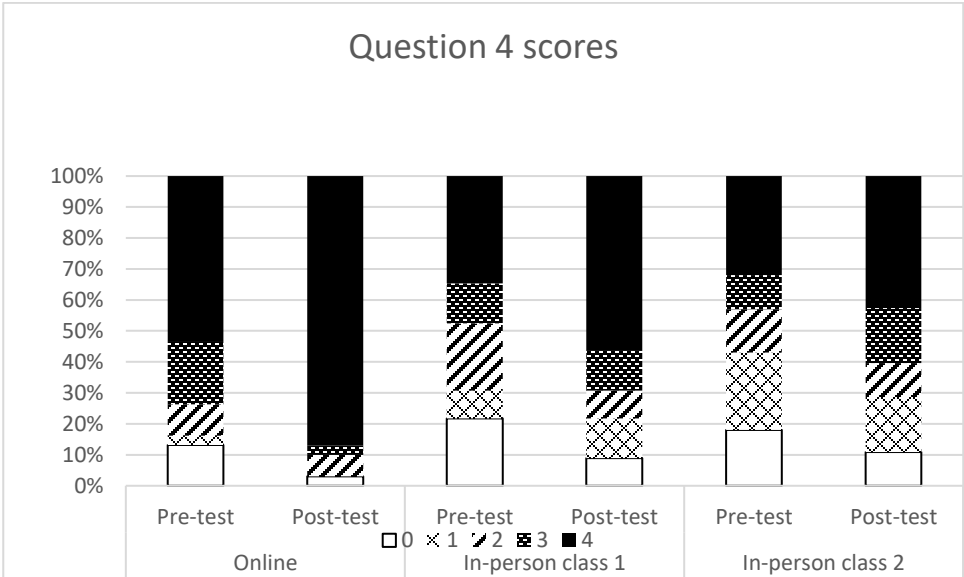
Question 4d provided a situation of far transfer from the CQS involving three successive measurements of incompatible spin components, e.g.,  $S_x \rightarrow S_z \rightarrow S_x$ . Performance on this question, which has no direct analogue in the CQS, was generally lower than for many other questions, which is not surprising. That said, students who provided correct answers consistently pointed to how each measurement of a particular component of spin “destroys” knowledge of the other spin components (which was an idea emphasized in the CQS), so that the two measurements of  $S_x$  will not necessarily match each other. Very few students on the post-test answered that the two measurements of  $S_x$  should match in this situation.

For all parts of question 4, a common incorrect response involved using the original state (not the collapsed state following the measurement of  $S_z$ ) to respond to the questions. This was seen on both the pretest and the post-test, somewhat less frequently on the latter. It is possible that such students either did not notice that the state should have already collapsed, or did not think such information was relevant. Prior research shows that some students think that a quantum state will return to its original state after enough time (even if it collapsed into another state); that measurement does not affect the state (which is only true for the eigenstate of the operator corresponding to the observable measured); or that discrete observables can change drastically between measurements (as opposed to continuous variables); these ideas could have also played a role [34,44,46]. Additionally, for questions 4c and 4d, some students who struggled appeared to, for instance, use the given state  $|\pm z\rangle$  to incorrectly conclude that the probability of measuring a particular value for  $S_x$  is zero, not realizing that  $|\pm z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle \pm |-x\rangle)$ .

Question 4 as a whole was designed to probe students' knowledge on the collapse of quantum states when specific spin components are measured. Each successive part goes deeper into a hypothetical situation in which observables are measured whose corresponding operators either do or do not commute. Students who, for instance, provide the correct state after the first measurement of  $S_x$  may not recognize that this is the state in which the next measurement must be made, or that after an intervening measurement of  $S_z$  is made, this system may not yield the same outcome if  $S_x$  is then measured again. It is illustrative to consider students' success and improvement in answering all four parts of the question. This is shown in Figure 4.3. It is clear that, on the post-test, students who answered three or four of the parts correctly increased in number, reaching 50% of the class or higher, while the number of students who correctly answered only two, one, or none of the parts decreased.

**Table 4.4 A table showing the shifts in distributions for all parts of question 4 from the pretest to the post-test for each class.**

Question 4 scores		0	1	2	3	4
Online	Pretest	13%	3%	10%	20%	53%
	Post-test	3%	0%	7%	3%	87%
In-person class 1	Pretest	22%	9%	22%	13%	35%
	Post-test	9%	13%	9%	13%	57%
In-person class 2	Pretest	18%	25%	14%	11%	32%
	Post-test	11%	18%	11%	18%	43%



**Figure 4.3 A bar chart showing the shifts in distributions for all parts of question 4 from the pretest to the post-test for each class.**

**4.4.6 Preliminary investigation of difficulties with expectation values**

Question 5 was added to the second in-person class to investigate students’ understanding of the expectation value of an observable, which prior research [60] has suggested is a challenging concept. In this administration during the second in-person class, students were additionally asked to calculate the expectation value of  $S_x$  in a given state. A wide range of responses appeared on the pretest; overall, the inconsistency in these responses, as well as the number of students who

left the question blank or answered “I don’t know,” indicates that students were not confident in their knowledge of expectation values.

The following responses on the pretest were present on the post-test in highly reduced numbers. One pattern of response was to list the eigenvalues that could result from a measurement made in the given state, sometimes alongside the probabilities with which they could be measured. Other students responded by correctly observing that any measurement made in the same initial state would yield results with the same probability distribution, but without showing any attempt at calculating the expectation value itself. Students in both groups were given no credit, since a different question had already asked them to provide the outcomes and probabilities of measuring each outcome in a given state. This is reminiscent of students’ answers to a question, in a separate study, about measurement uncertainty. When asked to calculate a nonzero uncertainty in the measurement of a particular observable, some students appealed to the fact that the given state had a nonzero chance of yielding either outcome, but did not proceed to calculate a numerical value for the uncertainty [122]. Students providing such responses when asked to calculate the expectation value may not know or remember what an expectation value is, and are only able to produce part of the relevant knowledge. On the pretests and post-tests, some responses to the expectation value question had phrases such as “expected value,” seeming to interpret the question as another way of asking for the possible measurement outcomes (i.e., “what values would one expect from a measurement of this observable?”). These are some possible explanations for the types of responses that students provided with some frequency that identified the outcomes and the respective probabilities of measuring each one.

Some students on the pretest, and nearly all students on the post-test, chose a valid method to write or calculate the expectation value. The dominant methods were (i)  $\left(\frac{\hbar}{2}\right)P\left(\frac{\hbar}{2}\right) +$

$\left(-\frac{\hbar}{2}\right)P\left(-\frac{\hbar}{2}\right)$ , where  $P\left(\pm\frac{\hbar}{2}\right)$  represents the probability of measuring  $\pm\frac{\hbar}{2}$ , and (ii)  $\langle\chi|\hat{Q}|\chi\rangle$ .

Despite this, some students were unable to successfully complete the calculation. Most commonly, this was a result of choosing to calculate the expectation value via method (i), but simply finding incorrect probabilities of each measurement outcome. Students also chose method (ii) but were left confused on how to proceed. Such students received partial credit. These difficulties mirror some of those found in previous studies [34,42,60], and the multi-institutional QMFPS study found that 54% of undergraduate students and 57% of graduate students similarly struggled with finding a correct expression for expectation value [70].

All these difficulties were observed in much smaller numbers on the post-test. Furthermore, the fact that many of the difficulties found in previous work, e.g., Ref. [60], appear to have been avoided could be a sign that the CQS is helpful in guiding many students toward a productive conception of expectation value, or at least a fluency in calculating expectation values that could be useful in further consolidation of knowledge with continued instruction. Though the data are from a single year of administration of CQS, we find that students may need help beyond traditional lecture-based instruction in achieving facility with expectation value for two-state systems. Furthermore, we are optimistic that the CQS is beneficial as a vehicle for enabling students to acquire such facility.

A summary of the difficulties found, along with the CQS number and the extent to which the average student performance improved from the pretest to post-test, is shown in Table 4.3.

**Table 4.5 Student difficulties addressed by the CQS questions, which are found in Appendix C.2. For each difficulty, previous studies are cited which have reported such observations.**

Difficulties	CQS #	Pre-/post-test #   comments
Action of an operator corresponding to an observable on a state confused with measurement [1,34,42,44,127]	1.1, 1.2, 1.3	1, 2a   Major improvement, but still difficult

Normalization of a quantum state [46,127]	1.3, 2.1, 2.3, 3.1, 3.2	2b, 4a   Students performed well on normalization questions in isolation (high correctness), but are less adept at providing a normalized collapsed state after a measurement (medium improvement)
Measurement basis is different from given basis [46]	2.2, 3.2	2c   Some improvement
Outcomes and probabilities of measurement outcomes for an observable, measurement basis matches given basis	1.3, 2.1	3   High pretest and post-test correctness
Measurement of various spin components in immediate succession [34,44,46,83]	3.1, 3.2	4b-4d   Some improvement
Difficulties with expectation values [34,42,60] (preliminary results)	4.1, 4.2	5   Major improvement

#### 4.4.7 Comparisons between instructors and learning environments

The two in-person classes with different instructors have quite similar profiles on pretest and post-test performance. Final scores on the post-test match closely when examined question-by-question, with the largest discrepancies found in questions 1 and 4b (see Table 4.2). It is unclear why, for the second in-person year, performance on questions 1 and 2a is so different, as both are intended to target similar concepts, but it is possible that the additional explanation that question 2b provides in interpreting the action of an operator on a quantum state was enough to cue these students into rejecting the statement, while accepting the statement provided in question 1 without giving it much further thought. For question 4b, the most common incorrect answers were discussed earlier in the subsection “*Normalization of a quantum state*,” and most of the students who provided such answers were in in-person class 2. Other moderate differences between the in-person years may be due to differences in instructors’ approaches to the material, or the way

students understood the material or the questions, but as a whole, there are not many such differences. That the instructors used different textbooks that are usually tuned to different approaches to teaching QM (although both instructors covered two-state systems early in the course) also suggests that the CQS is flexible enough to remain robust in a variety of QM curricula backed by different course textbooks.

On the other hand, comparing the online class with in-person class 2, which is the most robust comparison of the two environments because the instructor was the same, it may appear surprising that students performed largely better on the post-test in the online class despite not having access to collaborative learning opportunities. However, research [130,131] suggests that student behavior can differ depending on many factors, including stakes, between online and in-person assessments. In light of the consistency across the two in-person classes, it may be the case that the online scores are inflated.

In particular, we acknowledge that some students may have been able to consult resources that they were not intended to access during the online-administered pretest and post-tests. Even though, much like a closed-book and closed-notes quiz in an in-person class setting, the online pretests and post-tests were administered synchronously with students submitting their work at the end of the allotted time, the situations are still not quite the same, especially when all students had their cameras off. There was no way to adequately determine or enforce which resources they used during the examination time. As such, the results from the in-person classes may be regarded as more representative of performance without consulting any resources, with the pretests and post-tests administered in a more controlled environment. Another explanation, not mutually exclusive, is that the increased availability of class materials such as recorded lectures during online classes

could have given students a wealth of material to study from in preparation for the pretest and post-test.

Regarding the emergency remote teaching arrangements of the online class, while data do not exist for retention of concepts in the same capacity as in the second in-person class (discussed in the “*Retention and further learning after solutions were made available*” section that follows), the final overall grade distributions were comparable between the online class and in-person class. Though not a perfect measure, this offers some assurance that the two classes were not extremely different in the overall level of student knowledge and learning of QM. These findings suggest that the CQS offers meaningful benefits for all the classes in which it was implemented, even with their very different course structures and social backdrops.

#### **4.4.8 Retention and further learning after post-test solutions were made available**

Students were provided neither the CQS questions nor the pretest questions as resources before administration of the post-test; however, after students were graded on their post-test performance, the post-test solutions were posted on the course learning management system. We find that students performed very well on similar questions that were asked two weeks later on the midterm examination of the second in-person class. The post-test questions that had very similar analogues on the midterm exam were 2a, 2b, 2c, 3, 4a, and 4b. On these questions, correctness was nearly 100% for each, a heartening sign that almost all the students are getting what they should out of the overall instruction after having struggled productively on the pretest, CQS, and post-test. One question merits special attention. Question 2c, which assesses whether students would be able to deduce the need to change the basis before providing the probability of measuring a particular outcome, focuses on a concept that appeared on the post-test as both a nearly exact

reproduction and as a prerequisite for another question. In isolation, when explicitly asked to change the basis, students were able to do so at a nearly perfect rate, 93%, but when implicitly required as one step of another problem related to measurement, 76% of students recognized that a change of the basis was necessary to solve the problem. This suggests that, while students may understand that they must apply this concept when it is the primary focus of a question, they may not yet recognize its applicability when this concept is applicable but the question does not explicitly ask for it. For example, they may not check whether the state is written in the appropriate basis or transform to the appropriate basis when asked about outcomes of measurement, particularly when other features are present (see Table 4.4).

We note that prior research in introductory physics and quantum mechanics suggests that students who self-diagnose and evaluate their mistakes on earlier problem-solving tasks (e.g., quizzes or exams) are likely to do significantly better on those concepts on future exams, but that the act of providing solutions to students alone does not trigger such self-diagnostic behavior. This is true even if the students know that the material could show up on future testing, and the questions asked are identical [52,132]. Without explicit grade incentives and encouragement, many students tend not to learn from their mistakes by comparing their work to that presented in the provided solutions. Indeed, it was only when students were given an explicit reason to do this (e.g., being given a grade incentive for correcting their mistakes) that their performance on the second assessment improved.

We hypothesize that the time dedicated to struggling with the concepts may also be a powerful way to enable students to productively engage with the relevant material ahead of the exam. It is possible that experience with productive struggle in collaborative learning during the CQS activity cued students into a learning mode, making them attuned to their mistakes, and

motivated to clear up their difficulties. As a complementary hypothesis, students were in a position to better understand and make use of their available resources (which included post-test solutions) after engaging with the CQS than they would have been without it. In particular, students appear to have retained or learned the measurement concepts well between their post-test and midterm exam weeks after they had gone over the relevant material during class.

**Table 4.6 Student performance on similar questions given on a midterm exam about two weeks after the post-test, for in-person class 2 ( $N = 28$ ). The concept covered by question 2c appeared in two separate questions, each in a different context.**

Question #	Post-test	Midterm	Normalized gain	Effect size
2a	68%	93%	0.78	0.65
2b	96%	100%	1.00	0.27
2c	86%	93%	0.50	0.23
		76%	-	-
3	89%	96%	0.67	0.32
4a	71%	79%	0.25	0.16
4b	75%	100%	1.00	0.80

## 4.5 Conclusions

Validated clicker question sequences can be effective tools when integrated with classroom lectures. We developed, validated, and found encouraging results from implementation of a CQS on the topic of quantum measurement for two-state systems, in both online and in-person settings. We drew two comparisons among the implementations, one between two different instructors for identical modes of instruction (in-person), and one between different modes of instruction (online vs. in-person) for the same instructor. Post-test scores improved for nearly every question in each implementation, with exceptions typically being questions that had very high pretest performance to begin with. Our student interviews here and in past work [1,34,42,44,46,60,70] have indicated

that students have many conceptual difficulties with the measurement concepts examined here. By building on this work to specifically address these difficulties, the evidence of improvement after CQS instruction is likely to be due to learning rather than simply memorizing the correct answers.

In summary, it appears that the CQS provided noticeable and meaningful benefits in all observed cases despite differences in modality and instructors' lecturing styles. Within the ICAP framework, both interactive (in-person classes with peer interaction) and constructive (online classes without peer interaction) modes appear to be beneficial for students. In particular, it greatly reduces widespread student difficulties, e.g., confusing the action of an operator corresponding to an observable on a state with the act of quantum measurement. For in-person class 2, the data from a midterm exam given later in the semester indicate that students further strengthened their understanding of these concepts in the intervening time period when solutions to the post-test were provided, likely owing to the high levels of engagement and productive struggle that they experienced during the collaborative learning and class discussions of the CQS. Clearly, though, modality and instructor have some effects, whose nature will be investigated in future studies. In particular, while the CQS was useful in all of these cases, it is possible that these factors along with the choice of textbook could explain differences not only in the post-test but also in the pretest.

#### **4.6 Acknowledgments**

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## **5.0 Challenges in addressing student difficulties with measurement uncertainty of two-state quantum systems using a multiple-choice question sequence in online and in-person classes**

### **5.1 Introduction**

Measurement uncertainty in quantum mechanics (QM) is a foundational concept that has no classical analogue. Quantum measurement uncertainty is illustrated by the fact that when an observable is measured in a quantum state that is not an eigenstate of the corresponding Hermitian operator, measurement outcomes are not certain. In particular, the measurement collapses the state to one of the eigenstates of the operator corresponding to the observable with a certain probability, and if many measurements of this observable are conducted in an ensemble of identically-prepared systems, the standard deviation of those measurement outcomes is the measurement uncertainty. Furthermore, any subsequent measurements of the same observable made in a collapsed state, assuming no time-evolution, will yield the same outcome with 100% certainty. An observable is said to be well-defined when the quantum system is in an eigenstate of the corresponding operator. Also, the uncertainty principle states that if two observables correspond to operators which do not commute, then they cannot both be measured with 100% certainty, i.e., cannot both be well-defined, in the same quantum state. Because the uncertainty principle can be challenging for students, instructional resources have been developed to help students learn this concept in different situations. Measurement uncertainty and the uncertainty principle are fundamental tenets of quantum theory that are relevant in any context involving successive measurements of different observables, including fields of active research, such as the growing field of quantum information science.

Prior research suggests that students in quantum mechanics courses often struggle with many common difficulties [1,2,9–13,19,24,44,46,108–111,133], including the basic formalism [3,34], notation [14], wavefunctions [15,37], the nature of probability [4], measurement [3,7,15,42], and transferring learning from one context to other contexts [1,33]. For such difficulties as those described, research-validated learning tools can effectively help students develop a robust knowledge structure [5,6,16,17,70,112]. For example, our group has developed, validated and implemented Quantum Interactive Learning Tutorials (QuILTs) with encouraging results on many topics in QM, including quantum measurement of physical observables [43,55,60] and the uncertainty principle and Mach-Zehnder interferometer [36,51]. Other commonly used learning tools in physics include clicker questions, first popularized by Mazur [86] using his *Peer Instruction* method. These are conceptual multiple-choice questions presented to a class for students to answer anonymously, typically individually first and again after discussion with peers, and with immediate feedback [86]. They have proven effective and are relatively easy to incorporate into a typical course, without the need to greatly restructure classroom activity or assignments [105].

While these questions can be successfully implemented without additional technological tools, this research used an electronic response system, generally referred to as “clickers,” which automatically tracked student responses in real time. When presented in sequences of validated questions, clicker questions can systematically help students with particular concepts that they may be struggling with. Previously, such multiple-choice question sequences, or Clicker Question Sequences (CQS) related to several key QM concepts have been developed, validated and implemented [71,72,106,115,73,134]. Furthermore, previous work has been conducted to investigate student difficulties with the uncertainty principle as it applies to wavefunctions [46] as

well as two-state systems [135], but there has not yet been a documented effort to leverage the CQS method to address those difficulties. Here we describe the development, validation, and implementation of a CQS intended to help students learn about measurement uncertainty as it pertains to two-state quantum systems, and we discuss difficulties in identifying well-defined observables in a given state, calculating measurement uncertainty, successive measurements of various spin angular momentum observables, and other difficulties that naturally came up during implementation.

## **5.2 Methodology**

### **5.2.1 Development and validation**

The CQS on quantum measurement uncertainty is intended for use in upper-level undergraduate QM courses. During the development and validation process, we took inspiration from some of the previously-validated learning tools, including determination of learning objectives. In particular, much research involving cognitive task analysis, from both student and expert perspectives, has already been conducted in the development and validation of a QuILT and CQS on measurement uncertainty and the uncertainty principle for wavefunctions (including in the context of the orbital angular momentum), as well as a QuILT on the basics of spin-1/2 systems [36]. Ten student interviews had been conducted using a think-aloud protocol in the development of each of these learning tools, and the insights on student difficulties with regards to spin-1/2 systems and uncertainty principle for orbital angular momentum helped to guide the development of this CQS. In addition, we recently conducted four additional think-aloud interviews with physics

post-graduate students with this CQS and, among other things, found that they appreciated the flow of CQS questions in terms of how well they build on each other.

We adapted some of those questions while also drafting and iterating new ones for measurement uncertainty related to two-state quantum systems. To ensure that the material could be completed in the allotted class time, while offering maximal value to students, we prioritized the coverage of conceptual knowledge, used common difficulties as a guide, provided checkpoints that could stimulate useful class discussions, and avoided burdensome calculations. We iterated the questions many times amongst ourselves and with other faculty members to minimize unintended interpretations and ensure consistency and simplicity in terminologies and sentence constructions.

We aimed to address common stumbling blocks and emphasize key features that students may have missed in the large information content of a typical lecture. Some questions in the CQS employ a complex multiple-choice question format, in which students are presented a number of options (e.g., options I, II, III) and must select one of several choices (e.g., choices A, B, C, etc.) that consist of a subset of those options. Though this may increase student cognitive load, we and others in physics education research have successfully used such question models for their relative parsimony in addressing multiple relevant facets of a concept at once, with respect to the limited class time. This is also helped by the immediate feedback and scaffolding support students receive during class discussions of the questions.

The seventeen questions in the final version of the CQS focused on the following four learning goals: identifying well-defined observables in a given quantum state (4 questions), calculations related to measurement uncertainty and the generalized uncertainty principle (5 questions), features and commutation relations of the spin operators  $\hat{S}^2, \hat{S}_z, \hat{S}_x,$  and  $\hat{S}_y$  (5

questions), and compatible vs. incompatible observables (3 questions). (In the CQS, the words “compatible” and “incompatible” are used to describe observables and their corresponding operators interchangeably.)

### **5.2.2 Implementation**

The data presented here are from administration in a mandatory first-semester junior-/senior-level QM course at a large research university in the United States. Given that a physics course was the object of study, we elected to use quantum physics-framed language rather than quantum information-framed language to discuss these concepts. The final version of the CQS was implemented in two consecutive years, one online and one in person. In each instance, the instructor dedicated between two and three consecutive class sessions, each 50 minutes long, to complete the pre-test, CQS, and post-test.

During the online implementation, the CQS was presented as a Zoom poll while the instructor displayed the questions via the “Share Screen” function. For the in-person implementation, the poll was replaced by a functionally similar classroom clicker system. For each question, the instructor displayed the results after all students had voted, before a full class discussion of the validity of the options provided. Some of the questions involved more calculation than is typical of the conceptual questions in Mazur’s method for introductory courses, which are intended to take roughly a minute each. Instructors were urged to use their judgment in giving the class more time to answer such questions.

Because of difficulties in adapting to the online environment in a way that remained conducive to small-group student discussion, the Peer Instruction feature was largely forgone in

the online administration, but was realized in full for the in-person administration. We note also that the instructors were different for the online and in-person classes.

Table 5.1 below summarizes the learning goals, and the CQS questions and pre-test and post-test questions that cover these concepts.

**Table 5.1 Details of learning goals and their correspondence to questions on the CQS and the pre-test and post-test.**

Broad learning goals	Details of learning goals	CQS questions covering the concept	Pre-test/post-test questions covering the concept
Identify well-defined observables in a given state	Observables are well-defined in an eigenstate of the corresponding operator	1.1, 1.2, 1.3, 2.1	1, 2, 3, 4a
Calculate various quantities related to measurement uncertainty	Definition of uncertainty	1.4, 2.2	1
	Calculating uncertainty	2.3, 2.4	4a, 4b
	Generalized uncertainty principle	1.4, 2.5	6
Describe properties of spin operators $\hat{S}^2$ , $\hat{S}_z$ , $\hat{S}_x$ , and $\hat{S}_y$ and their products, as well as features of measurements of the corresponding observables	Commutation relations of spin operators	3.1, 3.2, 3.3	3, 5a-5e
	Commuting operators share a complete set of simultaneous eigenstates; non-commuting operators do not	3.2, 3.3, 3.4, 3.5	3, 5b, 5c, 5d
	Results of measurement of $S^2$	3.4, 3.5	5e
Describe behavior of compatible operators and incompatible operators (in a general sense beyond the spin operators)	Compatible operators share a complete set of simultaneous eigenstates	4.1	4
	Incompatible operators do not share a complete set of simultaneous eigenstates.	4.2, 4.3	4

### 5.2.3 Assessment

To determine the effectiveness of the CQS, we developed and validated a pre- and post-test containing questions on topics covered in the CQS. The post-test was a slightly modified version of the pre-test, containing changes such as a shift from eigenstates of  $\hat{S}_x$  to eigenstates of  $\hat{S}_y$ , but otherwise remaining conceptually similar. In both online and in-person classes, students completed the pre-test immediately following traditional lecture-based instruction on the topic. After administration of the CQS over two to three class sessions, students completed the post-test. Two researchers graded the pre-test and post-test and, after discussion, converged on a rubric, for which the inter-rater reliability was greater than 95%.

During both online and in-person classes, the two to three classes were dedicated exclusively and consecutively to administration of the pre-test, CQS and ensuing discussions, and post-test. Since the CQS questions build on each other, they typically take less time to complete than if the questions were not part of a sequence. The only other content assigned over this duration was a traditional textbook homework set which overlapped with this period; given our past experience, we do not believe this had a significant effect on the more conceptual post-test performance. That said, this study is nonetheless quasi-experimental [128] in design, in light of these and other factors over which we did not have complete control.

The pre- and post-test questions are reproduced in Appendix D.1. Questions Q1-Q3 on the pre- and post-test provided students three possible answers from which to choose, and credit was awarded for correctly selecting or omitting each answer, for a total of up to three points per question. For these three questions, correct answers are bolded. For the free-response questions,

Q4a was out of one point, while Q4b and all parts of Q5 were scored with two points, one each for answer and reasoning. A more detailed breakdown of the questions is provided in the next section.

### 5.3 Results

The pre-test and post-test results for each question, as well as normalized gains [120] and effect sizes [121], corrected for small sample size through a multiplicative factor equal to

$\frac{N-3}{N-2.25} \sqrt{\frac{N-2}{N}}$ , are listed in Tables 5.2 (online with  $N = 27$ ) and 5.3 (in-person with  $N = 23$ ).

Overall, the results are encouraging, and students performed well on the post-test, with relatively high normalized gains and generally medium to large effect sizes. The multiple-choice questions (Q1-Q3) had reasonably high pre-test scores across both classes.

**Table 5.2 Results of the online administration of the CQS via Zoom. Comparison of pre- and post-test scores, along with normalized gain [120] and effect size as measured by Cohen’s  $d$  [121], for students who engaged with the CQS ( $N = 27$ ).**

Question #	Pre-test mean	Post-test mean	Normalized gain	Cohen’s $d$
<b>Q1</b>	78%	94%	0.72	0.86
<b>Q2</b>	94%	98%	0.60	0.31
<b>Q3</b>	95%	93%	-	-
<b>Q4a</b>	78%	85%	0.33	0.18
<b>Q4b</b>	52%	76%	0.5	0.67
<b>Q5a</b>	87%	93%	0.43	0.20
<b>Q5b</b>	69%	81%	0.41	0.37
<b>Q5c</b>	70%	83%	0.44	0.35
<b>Q5d</b>	37%	76%	0.62	0.85
<b>Q5e</b>	31%	48%	0.24	0.38

**Table 5.3 Results of the in-person administration of the CQS. Comparison of pre- and post-test scores, along with normalized gain and effect size as measured by Cohen’s  $d$ , for students who engaged with the CQS ( $N = 23$ ).**

Q#	Pre-test mean	Post-test mean	Normalized gain	Cohen’s $d$
Q1	80%	99%	0.93	0.98
Q2	91%	97%	0.69	0.29
Q3	83%	94%	0.67	0.36
Q4a	65%	96%	0.89	0.81
Q4b	35%	78%	0.66	1.01
Q5a	65%	96%	0.88	0.91
Q5b	52%	85%	0.68	0.78
Q5c	46%	80%	0.64	0.76
Q5d	48%	70%	0.42	0.47
Q5e	50%	80%	0.61	0.74

Table 5.4 presents a brief overview of the CQS data.

**Table 5.4 Summary of notable result trends in CQS responses across both years. For ease of comprehension, correctness rates for questions are referred to as low (0-40%), medium (41-70%), and high (71-100%). Note that a small number of questions administered in the in-person implementation did not appear in the online implementation, and vice versa.**

CQS #	Overall performance	Comments
1.1-1.4	Low to medium	<p>Students were only moderately successful at identifying well-defined observables, especially for less straightforward cases, e.g., in CQS 1.2.</p> <p>For CQS 1.3, the most popular distractor was the one that asserted that energy could not be well-defined for any state (where the Hamiltonian <math>\hat{H} = C(\hat{S}_x + \hat{S}_z)</math>).</p> <p>For CQS 1.4, many students considered <math>\frac{\hbar}{2}</math> to be the minimum product of uncertainties for any observables whose operators do not commute; some students also stated that quantum measurement uncertainty comes from apparatus imprecision.</p>

2.1-2.5	<p>Generally low to medium</p> <p>CQS 2.1 had medium performance online and high performance in-person</p>	<p>CQS 2.1 asked students whether uncertainties of specific observables were zero or nonzero, which most answered with reasonable success.</p> <p>For CQS 2.2, it was clear that most students knew the formulation <math>\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2</math>, but were less comfortable with <math>\sigma_A^2 = \langle (A - \langle A \rangle)^2 \rangle</math>.</p> <p>CQS 2.5 showed that students may have been unfamiliar with applying the generalized uncertainty principle to the spin operators, e.g., <math>\hat{S}_x</math> and <math>\hat{S}_y</math>.</p>
3.1-3.5	<p>Generally high</p> <p>Low for CQS 3.3</p>	<p>Students had an easy time with commutation relations of simple operators (e.g., <math>\hat{S}_x</math>), but tended to struggle somewhat more as the operators became more complicated (e.g., <math>\hat{S}_x \hat{S}_y</math>).</p> <p>For CQS 3.3 online, the instructor gave a hint reminding students of how to simplify commutation relations when operators are multiplied together. During the second vote, the correctness rate improved to medium.</p> <p>CQS 3.4-3.5 (in-person only) were meant to help students with the results of a measurement of <math>S^2</math>.</p>
4.1-4.3	<p>Medium to high</p>	<p>CQS 4.1, dealing with properties of observables corresponding to compatible operators, was more challenging for the students than the other questions in this cluster.</p> <p>4.2 (incompatible operators) and 4.3 (generalization of incompatible operators) had high correctness rates; these questions may have been more intuitive or less complicated.</p>

Below, we discuss some difficulties that were successfully addressed during the administration of the CQS for both years, as well as some that remained for smaller percentages of students.

### 5.3.1 Difficulties that were successfully addressed

#### 5.3.1.1 Identifying observables that are well-defined in a state

Questions Q1-Q3 on the pre- and post-test asked students to identify observables that are well-defined in a given state. Most students correctly selected  $S_x$  or  $S_y$ , identifying that the state is an eigenstate of the corresponding operator. In Q1, some students did not select  $S^2$ , which is also well-defined in the given state because its corresponding operator is proportional to the identity operator and commutes with  $\hat{S}_x$ . Some students incorrectly selected  $S_z$ , which may be due to the use of  $S_z$  in class as a frequent example where the Hamiltonian  $\hat{H} \propto \hat{S}_z$ . Questions such as CQS 1.1 and CQS 2.1 address these issues. On the post-test, the correctness rate on questions Q1-Q3 increased, with the exception of Q3 in the in-person administration, which had an especially high pre-test score (see Tables 5.2-5.3). Since pre-test scores on all three questions were quite high across both years, the normalized gains and effect sizes are not as informative, but this indicates that students have a relatively strong grasp of the concepts involved in these questions. This being said, Q1 had a large effect size for both implementations, indicating noticeable improvement.

#### 5.3.1.2 Calculating the measurement uncertainty

On the pre- and post-test, Q4 presented students with two states (Q4a and Q4b). The question asked them to determine whether the uncertainty in measuring a particular component of spin was zero in the given state (similar states involving  $x$ - or  $y$ -components of spin were provided in the pre- and post-test, respectively), and to calculate the uncertainty if it was not zero in the given state. Q4a provided a state in which the observable could be measured with 100% certainty, so students who indicated this received full credit. Across both implementations for Q4b, we

decided to give full credit to students who were able to provide the formulas for calculating uncertainty (e.g., an observable  $A$  has uncertainty  $\sigma_A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$  and the symbols under square root are the expectation values of  $A^2$  or  $A$ ). As the approach for solving the problem, and not the answer of the numerical calculation, was the primary learning goal assessed by this question, our rubric thus avoided penalizing students who made a mathematical error in subsequent steps. Also, some students correctly identified whether the measurement uncertainty is zero or non-zero, but justified their answer only by invoking the probabilities of measuring each possible outcome. These students received half credit based upon our rubric. Questions CQS 2.3 and 2.4 were added for the in-person implementation to address issues related to the calculation of measurement uncertainty, and while Q4a had a high effect size in only the in-person implementation in part due to lower pre-test scores, there were reasonably impressive normalized gains and effect sizes for Q4b across both years (see Tables 5.2-5.3).

### 5.3.1.3 Results of successive measurements of identical or non-commuting observables

On the post-test, Q5 asked students for the final outcome of consecutive measurements of some permutation of the observables  $S_x$ ,  $S_y$ , and  $S^2$ , specifically testing whether students could recognize what happens when the measurements involved in the question corresponded to operators that did or did not commute with each other. Question Q5a asked about two consecutive measurements of  $S_x$ , Q5b about consecutive measurements of  $S_x$  and  $S_y$ , and Q5c about consecutive measurements of  $S_x$ ,  $S_y$  and  $S_x$  again. For Q5a, while most students correctly answered that the measurement in the collapsed state will yield  $-\frac{\hbar}{2}$  as the outcome, some students did not recognize that  $\hat{S}_x$  and  $\hat{S}_y$  do not commute. Thus, for question Q5b, they answered that the outcome of the  $S_y$  measurement would still be  $-\frac{\hbar}{2}$ , corresponding to  $| -y \rangle$ , and that in Q5c, the

final  $S_x$  measurement would yield  $-\frac{\hbar}{2}$ , corresponding to  $| -x \rangle$ , neither of which is correct. Questions such as CQS 2.5 addressed these issues, and in general, the post-test scores show reasonable improvement for all parts of Q5 during the online and in-person administrations (see Tables 5.2-5.3).

#### **5.3.1.4 Results of successive measurements of $S_x$ and $S^2$**

Question Q5d asked students for the final outcome of consecutive measurements of  $S_x, S^2$  and  $S_x$  again. On the pre-test, some students stated that for the final measurement of  $S_x$ , either eigenvalue  $\pm \frac{\hbar}{2}$  and eigenstate  $| \pm x \rangle$  could be obtained, and some explicitly cited  $\hat{S}_x$  and  $\hat{S}^2$  not commuting with each other for their reasoning. CQS 3.5 addresses measurement of  $S^2$  immediately after  $S_x$ , and CQS 4.3 and other questions in that sequence helped students generalize from spin-1/2 systems to more generic observables that correspond to operators that do or do not commute, and how such relationships may affect the measurements of those observables in a given quantum state. Student post-test performance showed better understanding of these concepts.

#### **5.3.1.5 Conflation of eigenstates and eigenvalues**

In the in-person implementation, on some questions only on the pre-test (not the post-test), some students wrote, e.g., that “The state will collapse into  $-\frac{\hbar}{2}$ ,” or “It is in the state  $-\frac{\hbar}{2}$ .” While the CQS did not explicitly focus on distinguishing between the collapsed state and measured value of the observable, it is likely that the precise language used throughout the CQS helped students distinguish between eigenvalues and eigenstates on the post-test.

## 5.3.2 Examples of persistent difficulties

### 5.3.2.1 Incorrect answers for the results of a measurement

Across both years' implementations, there were some students who on both the pre- and post-test answered that the result of a measurement was the eigenvalue multiplied by the eigenstate. For example, for question Q5a-e, they stated that making a measurement of  $S_x$  in the state  $|-x\rangle$  would yield an outcome of  $-\frac{\hbar}{2}|-x\rangle$ . They also claimed, e.g., that the outcome for Q5a would be  $\frac{\hbar^2}{4}|-x\rangle$ , the outcome for Q5b would be  $\pm\frac{\hbar^2}{4}|\pm y\rangle$ , and the outcome for Q5c outcome would be  $\pm\frac{\hbar^3}{8}|\pm x\rangle$ . This type of reasoning may be closely related to the student difficulty that an operator's action on a quantum state represents a measurement of the corresponding observable in the state and should be investigated further [46]. It is interesting that these students' answers remained the same on the pre-test and the post-test. This difficulty did not fall within the scope of this CQS, but it was addressed in another CQS which will be reported upon in a future publication. This may explain why the difficulty was observed to be rare but resistant for the duration of this CQS.

In the in-person implementation, another mistake was observed somewhat frequently even on the post-test. Question Q5d asked for the final outcome when  $S_x, S^2$  and then  $S_x$  were measured in that order in immediate succession, with the first measurement of  $S_x$  yielding  $-\frac{\hbar}{2}$ . Some students correctly stated that the intervening measurement of  $S^2$  did not collapse the state since the state was already an eigenstate of  $\hat{S}^2$ , but chose the *wrong* eigenvalue and eigenstate ( $+\frac{\hbar}{2}$ , corresponding to  $|+x\rangle$ ). These students were given full credit in recognition of their correct

reasoning. If this is reflective of a deeper difficulty rather than a careless mistake, we have no compelling speculation with regard to what that difficulty may be.

### 5.3.2.2 Conceptual difficulties and ambiguities regarding uncertainty relations

In the in-person implementation, we added an additional free-response question (not shown in Table 5.3) asking students to explain in their own words what it means for two observables, A and B, to have an uncertainty relation between them. Most students gave an answer involving the inability to know with full certainty the values of observables whose corresponding operators do not commute, but some framed their answers in terms of the position-momentum uncertainty relation (commonly cited as  $\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$ ), which is not related to two-state systems, rather than using the generalized uncertainty principle. As a result, these students noted that the measurement uncertainties of two observables must multiply to be greater than or equal to  $\frac{\hbar}{2}$ . These results support the findings of a previous study [135]. Though many questions in the CQS implicitly required knowledge of an uncertainty relation, only CQS 2.5 explicitly discussed the uncertainty principle and its applications to spin-1/2 systems, so this concept is worth emphasizing more in the future.

Another related response to this broad question about the uncertainty principle was “You can measure A or B, but not both.” This type of response was observed only in the pre-test, and not the post-test, indicating that these students may have realized the difference between the ability to physically measure an observable versus being able to predict with 100% certainty what outcome would be obtained when the measurement of an observable is made. Other students were not strictly incorrect, but were somewhat unclear in their qualitative responses, which included statements such as “If we know the value of A, then B is a point of complete uncertainty,” or “if

we know A with 100% certainty, we will have 0% certainty for B.” One interpretation of these responses is that “complete uncertainty” or “0% certainty” would refer to, in the case of a two-state system, an equal 50% chance of measuring either outcome. However, it is also possible that these students had the position-momentum uncertainty relation in mind, in which it is intuitive that a (continuous) decrease in uncertainty  $\sigma_x$  must be compensated for by an increase in uncertainty  $\sigma_{p_x}$  in order to maintain the product to be greater than or equal to  $\frac{\hbar}{2}$ . In the case of a two-state system, rather than refer to this as “complete” uncertainty, a more accurate description may be something like “maximum” uncertainty. (It is worth noting that, in a spin-1/2 system, the maximum uncertainty for any component of spin is exactly  $\frac{\hbar}{2}$ .)

Furthermore, in response to this question about the uncertainty principle between observables A and B, many students stated something more general to the effect of “we can never know both quantities exactly at the same time” if their corresponding operators did not commute. Additionally, many students noted that “measurement of one [observable] will possibly affect the other,” referencing the collapse to an eigenstate of the observable measured, in which the other observable would not be well-defined. Students were given full credit for all of these responses, since they have all articulated that the more is known about the value of one observable, the less is known about the value of another observable when their corresponding operators do not commute. While on the pre-test, some students were confused about uncertainty relations or left the question blank, nearly every student answered on the post-test in one of the ways discussed and thus earned full credit.

### 5.3.3 Comparisons between online and in-person implementations

Before comparing the online and in-person implementations, we note that some revisions were made to improve the CQS. These improvements affected the presentation of concepts that are covered in questions Q3, Q4b, and Q5d-e, mostly by providing more scaffolding to help students with solving these problems. The pre- and post-test questions remained nearly unchanged, with the exception of a clarification for questions Q5a-e as described in the next section. A one-to-one comparison can thus be drawn between the online and in-person implementations for pre- and post-test questions Q1, Q2, Q4a, and Q5a-c, as the CQS questions that covered the relevant concepts did not undergo any changes between years. For these questions, students' post-test performance does not differ appreciably between years. However, for the free-response questions Q4a and Q5a-c, the gap between pre-test and post-test performance is larger for the in-person class, as indicated by the larger normalized gains and effect sizes (see Tables 5.2-5.3).

For all questions on the pre- and post-test, the trend appears to remain similar: The average pre-test performance in the in-person implementation was, in general, somewhat lower, but post-test performance for both groups was comparable for all questions aside from Q5e, which is discussed in the next section. It is interesting that students performed about equally well on the post-test for both administrations, given that the online learning environment had greatly reduced opportunity for peers to discuss their responses with each other. We acknowledge that one possibility is students' ability to consult resources, despite being instructed not to do so, during the online-administered pre- and post-tests. Even though students were told that the quizzes were closed-book and closed-notes, such a rule could not be enforced when, as was the case, most students had their cameras off. Even so, those students would have had access to the same resources during both the pre- and post-test, so the sizable improvements in the post-test scores of

the online class are still a good sign of the benefits of the CQS. We also observed that many more students left some pre-test questions completely blank in the in-person implementation compared to the online implementation, despite both classes having received the same amount of time to complete the pre- and post-tests. Since the students were given sufficient time to complete the pre- and post-tests, the cause of this higher occurrence of leaving some questions unanswered during the in-person class is unclear. This may be due to students not feeling confident enough to answer, or dealing with additional test anxiety or apathy not experienced by the students in the online implementation. In particular, while online classes have their disadvantages, there were also some benefits and conveniences that would have been lost in the transition back to in-person classes, which could have contributed to this phenomenon. Finally, as we have noted before, the different instructors between years could have also been a factor; e.g., during the online administration, it is possible that more emphasis was placed on content related to two-state spin systems.

The administration of the CQS in an online learning context may have affected student performance differently as compared to the in-person administration. However, regardless of whether the performance across years can be compared one-to-one, it is clear that the CQS has had a beneficial impact on student learning for both the online and in-person implementations.

### **5.3.3.1 Result of measurement of $S^2$**

For the in-person implementation, CQS 3.4-3.5 were added to provide additional scaffolding on concepts relating to the observable  $S^2$ . Although the two implementations therefore cannot directly be compared with regard to student learning of these concepts, there are some differences worth examining. During the online implementation of the CQS, students demonstrated difficulties with the outcomes obtained from a measurement of  $S^2$ . This is relevant in post-test question Q5e, which asked students to provide the outcomes of successive

measurements of  $S_x$ ,  $S_y$ , and  $S^2$  in a state, in that order. During online administration, this question had a large variety of responses that were difficult to score, but were still useful in shedding light on student difficulties. In particular, some students did not recognize that the eigenvalues obtainable from a measurement depend on the observable being measured. As an example,  $\hat{S}_z$  and  $\hat{S}^2$  share eigenstates, but a measurement of  $S_x$  made in the state  $| -x \rangle$  would yield  $-\frac{\hbar}{2}$ , whereas a measurement of  $S^2$  in that same state would yield  $\frac{3}{4}\hbar^2$ . However, multiple students answered that, should the measurements of  $S_x$  and  $S^2$  be made in succession in this state, they would both yield eigenvalues of  $-\frac{\hbar}{2}$ . One possible reason for this type of response is that these students had associated the label  $| -x \rangle$  primarily with the eigenstate of operator  $\hat{S}_x$  with eigenvalue  $-\frac{\hbar}{2}$ , so although some students realized that this state is a simultaneous eigenstate of  $\hat{S}^2$ , they had difficulty with the corresponding eigenvalue (the state  $| -x \rangle$  also carries an implicit label for the quantum number  $s$ , which is dropped for ease of notation, which may contribute to this difficulty).

Question Q5e also revealed that some students may not have realized when answering the question that the only possible measured eigenvalue of  $S^2$  in any two-state spin system is  $\frac{3}{4}\hbar^2$ , as the only eigenvalue of  $\hat{S}^2$  is  $\hbar^2 s(s+1)$ , with  $s = \frac{1}{2}$  for spin-1/2 systems. When listing the possible eigenvalues for the measurement of  $S^2$ , in addition to the previously-mentioned  $-\frac{\hbar}{2}$  (and the closely related  $\frac{\hbar}{2}$ ), other answers included  $\pm \frac{\hbar^2}{4}$  and  $-\frac{3}{4}\hbar^2$ , which are not unreasonable responses. The response  $\pm \frac{\hbar^2}{4}$  may come from literally squaring the eigenvalues of, e.g.,  $\hat{S}_x$  while matching the sign of the eigenstate label  $| \pm x \rangle$ , in analogy with the eigenvalues of  $\hat{S}_x \pm \frac{\hbar}{2}$  being associated with the respective eigenstates  $| \pm x \rangle$ . Finally, the response  $-\frac{3}{4}\hbar^2$  may be analogous to

the notion that the state  $| -x \rangle$  is associated with a negative eigenvalue  $-\frac{\hbar}{2}$  for a measurement of  $S_x$ .

In the online administration, the CQS did not explicitly address measurements of the observable  $S^2$ . Therefore, it is not surprising that students were left with some alternative conceptions. As a result of this, we refined the CQS to explicitly address the eigenvalues and eigenstates obtained from a measurement of  $S^2$  in the following in-person implementation. On Q5a-e of the pre- and post-test for the in-person implementation, we made the small addition of explicitly asking students to provide both the eigenvalue and eigenstate resulting from the measurements posed, instead of simply asking “What is the result of the measurement?” as in the online administration in the preceding year (see Appendix D.1 for details). Previously, for the online implementation, students were given credit for providing either the obtained eigenvalue or the eigenstate after the measurement. During the in-person implementation, as a result of the change, students provided less ambiguous answers to these questions. Though the differences in post-test performances for online and in-person administrations could be attributable to more than one factor, the additional scaffolding provided by CQS 3.4-3.5, which specifically addressed these concepts, appears to have been effective. Across the two years, students’ performance rose from 48% percent in the online implementation to 80% on the in-person implementation (see Tables 5.2-5.3).

A summary of student difficulties observed in the pre- and post-tests is presented in Table 5.5.

**Table 5.5 Student difficulties addressed by the CQS questions, which are found in Appendix D.2.**

Concept	Observed student difficulties	CQS #	Pre-/post-test #   comments
Identifying observables that are well-	Not realizing that an eigenstate of e.g., $\hat{S}_y$ is also an eigenstate of $\hat{S}^2$ ; stating that $\hat{S}_z$ is well-defined in an eigenstate of $\hat{S}_y$	1.1, 1.2, 1.3, 2.1, 2.5	Q1, Q2, Q3, Q4a   Some improvement

defined in a state			
Calculating measurement uncertainty	Using nonzero probability of measuring each outcome to justify nonzero uncertainty rather than actually calculating uncertainty	2.2, 2.3, 2.4, 2.5	Q4b   Major improvement
Results of successive measurements of observables whose corresponding operators are compatible or incompatible	<i>[For in-person, many students did not provide both the eigenvalue and the state on the pre-test, but did provide both on the post-test]</i>	3.1, 3.2, 3.3, 4.1	Q5a   Some improvement for online; major improvement for in-person
	Stating that successive measurements of incompatible observables (e.g., $S_x$ and $S_y$ ) will yield the same eigenvalue or not change the state	3.1, 3.2, 3.3, 4.2, 4.3	Q5b, Q5c   Some improvement for online; major improvement for in-person
	Stating that successive measurements of compatible observables (e.g., $S^2$ and $S_x$ ) will change the state	3.1, 3.2, 3.3, 3.4, 3.5, 4.1	Q5d   Major improvement for online; some improvement for in-person
Conflation of eigenstates and eigenvalues	Stating, e.g., that a measurement will “collapse the state into $-\frac{\hbar}{2}$ ”	3.4, 3.5	Q5a-e   Major improvement (in-person only)
Conceptual difficulties regarding uncertainty relations	Stating, for example, that if operators $\hat{A}$ and $\hat{B}$ (corresponding to observables $A$ and $B$ ) do not commute: <ul style="list-style-type: none"> <li>• <math>\sigma_A \sigma_B \geq \frac{\hbar}{2}</math> for generic observables <math>A</math> and <math>B</math></li> <li>• Observables <math>A</math> and <math>B</math> cannot both be measured (often unclear whether referring to the measurements themselves or the precision of the measurements)</li> <li>• Knowing <math>A</math> implies infinite uncertainty for <math>B</math></li> </ul>	1.4, 4.3	Q6   Major improvement (in-person only)
Result of measurement of $S^2$	Giving an incorrect eigenvalue for a measurement of $S^2$	3.4, 3.5	Q5e   Some improvement for online; major improvement in-person

## 5.4 Summary

Validated CQS can be effective tools when implemented alongside classroom lectures. We developed, validated, and found encouraging results from implementation of a CQS on the topic of measurement uncertainty in two-state quantum systems, in both online and in-person settings. Post-test scores improved for every question following the administration of the CQS, with the exception of Q3 in the online implementation; this question tested students on whether energy can be well-defined for various Hamiltonians, and had an exceptionally high pre-test score. While the performance on the multiple-choice questions was high to begin with on the pre-test, there was significant improvement in the free-response questions. Effect sizes varied for the online implementation, but notably were large for nearly every free-response question in the in-person implementation, since the pre-test scores were lower than in the online implementation. This difference in pre-test performance could be reflective of differences in student preparation or behavior, or other factors such as instructor or environment. An examination of the post-test scores in the online and in-person implementations shows comparable performance in both years on most questions, demonstrating the effectiveness of the CQS in both administrations.

## **5.5 Ethical statement**

This research was carried out in accordance with the principles outlined in the University of Pittsburgh Institutional Review Board (IRB) ethical policy. Informed consent was obtained from all interviewed students who participated in this investigation.

## **5.6 Acknowledgments**

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## 6.0 Student Understanding of the Bloch Sphere

### 6.1 Introduction

Quantum information science and engineering (QISE) is an exciting interdisciplinary field that has applications in quantum computing, quantum communication and networking, and quantum sensing, which are attractive to scientists and engineers for many reasons. Computer scientists and engineers are developing quantum algorithms for various problems, including ones that become impractical for classical computers to solve at large scales. For example, on a classical computer, the problem of factoring products of large prime numbers scales exponentially with the size of the prime numbers, but on a quantum computer utilizing Shor's algorithm, the problem scales roughly as a polynomial instead. For future applications in science, physicists and chemists are also excited about the potential of quantum computers to solve important problems in their disciplines in which solving the Schrödinger equation plays an important role. The development of robust quantum bits (qubits) and scalable quantum computers demands the expertise of physicists and engineers alike. For all these reasons and more, this area of study holds a great amount of promise for students from many science and engineering disciplines who are interested in careers in QISE-related fields [28,29].

One teaching tool for the introduction of quantum states and their visualization is the Bloch sphere, which allows for visualization of the states of a qubit, the fundamental functional unit of a quantum computer. It can be an important and powerful aid for understanding the properties of two-state systems, but students often have difficulties with it. Additionally, the Bloch sphere is a

highly useful tool for current research, including in quantum sensing and tomography, and experimentalists in the field routinely use it to characterize a single qubit in their work. The Bloch sphere allows one to graphically understand a single-qubit state, including mixed states via the density matrix, and the operations that can be done through single-qubit gates.

Here we describe the development, validation and implementation of a research-based tutorial on the Bloch sphere. The course in which the Bloch sphere tutorial was implemented is an interdisciplinary course involving students from many fields. The Bloch sphere is taught as a foundational topic in this course, which was the only mandatory course in a new “Foundations of Quantum Computing and Quantum Information” undergraduate certificate program at a large research university in the United States.

The Bloch sphere is a mathematical mapping of a complex two-dimensional Hilbert space onto the surface of a unit sphere in 3-D real space, which does not have any intrinsic mapping onto 3-D physical space. The Bloch sphere brings an additional geometric interpretation to what is otherwise a linear algebra-heavy topic, which is valuable in providing more context, additional perspectives, and intuitive representations to the concept of a qubit. States that require three independent real numbers to describe in two-dimensional Hilbert space, if one considers normalization (i.e.,  $(a_1 + ia_2)|0\rangle + (a_3 + ia_4)|1\rangle$ , where  $\sum a_i^2 = 1$ , where  $a_i$  are real numbers), only need two to be located on the Bloch sphere: the angles  $\theta$  and  $\phi$ . This is accomplished by rephasing any state such that the  $|0\rangle$  component is real and non-negative; such a state can be expressed in the form  $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ . This Bloch sphere tutorial helps students learn the definitions, conventions, and parameters that are used to construct the Bloch sphere, and after laying down the basics, it helps students explore various visual interpretations of the Bloch sphere. Points on the surface of the Bloch sphere represent pure states, while points inside the Bloch sphere

represent mixed states; though mixed states are outside the scope of our targeted introductory level, familiarity with this basic Bloch sphere representation that we focus on in the tutorial provides a foundation upon which additional concepts can be built in future instruction.

Prior research suggests teaching quantum mechanics (QM) can often result in common difficulties shared by a number of students, and moreover, that these difficulties can be mitigated or eliminated by well-designed research-validated learning tools [6]. For instance, others have worked on developing interactive activities [17,133,136], investigating and leveraging student models [10,24,112], developing visualizations [5,16,111], and finding pedagogical value in authentic problems [108,109]. Other investigations have been made into student difficulties on a general level [1,2,11,15,34], on specific topics [10,44], in undergraduate and graduate contexts [9,12,46,110], and by considering an epistemological perspective [13]. Difficulties include the basic formalism [3,34], notation [14], wavefunctions [15,37], the concept of probability [4], measurement [3,7,15,42], and transferring learning from one context to other contexts [1,33]. We have been researching student difficulties and using the research to guide the development of learning tools for concepts covered in undergraduate QM courses [70]. Previous work from our group includes Quantum Interactive Learning Tutorials (QuILTs) on topics such as the Mach-Zehnder interferometer and quantum key distribution, and Clicker Question Sequences (CQSs) on topics such as the basics and change of basis, quantum measurement, time-development, and measurement uncertainty of two-state quantum systems [51,75,115,122,137,138]. Even after traditional lecture-based instruction, students may not yet have a strong grasp of important concepts, but further engagement and practice using a research-validated tutorial may help them develop additional fluency with those concepts. To that end, we have developed and validated a tutorial to help students learn about the Bloch sphere.

Given the interdisciplinary nature of the field of quantum information science, for which this tutorial is intended, there is a need to standardize the language to be accessible and unambiguous for everyone regardless of background. Some trends have already taken hold in the field, including a distinction between interpreting phenomena “classically” as opposed to “quantumly,” rather than “quantum mechanically”; speaking of “measuring qubits” as opposed to measuring a physical observable; referring to a “measurement basis”; or measuring specific states such as  $|0\rangle$  or  $|1\rangle$  as outcomes rather than the corresponding eigenvalues. These are linguistic constructions that quantum physicists have typically not used. However, as there is a one-to-one correspondence, e.g., between the eigenvalues obtained by making a measurement of an observable in a specific state, and the eigenstates that are represented in the standard basis as  $|0\rangle$  and  $|1\rangle$ , it can serve as useful shorthand to say that a qubit is measured to be in the  $|1\rangle$  state to convey that a measurement made on the system yields the eigenvalue corresponding to the  $|1\rangle$  state.

In this research on the development, validation, and implementation of the Bloch sphere tutorial, we opt to use the prevailing terminology in the field so that students can become familiar with the language used by their textbooks, instructors, and other professionals in their studies. Not coincidentally, this language also serves to express many concepts more directly and in fewer words, and thus may reduce the cognitive load needed to learn them, as many of the physical details are immaterial to the contexts in which these students will apply the concepts that they learn. All that said, it is still crucial to maintain the integrity of language used to educate physicists who will be working to develop robust qubits and build real quantum computers, work that very much requires understanding the entire quantum physics taught in typical undergraduate and graduate physics courses.

## 6.2 Methodology

### 6.2.1 Development and validation

Student difficulties and potential responses to questions were explored through preliminary open-ended questions asked on an exam in an undergraduate QM course following traditional lecture-based instruction on the requisite concepts. Common student difficulties were identified based on these student responses. These difficulties were then used as a guide over the subsequent development of the tutorial. Four students were interviewed several times using a think-aloud protocol, spanning a total of roughly fifteen hours. Their feedback was used to gauge overall flow and whether the tutorial was at the appropriate level, as well as to identify blind spots, and their suggestions were incorporated into the subsequent versions of the tutorial. Throughout, the tutorial was repeatedly iterated with constant discussions among the authors, and with three additional faculty experienced in teaching QM and solid-state physics continually contributing feedback as well.

Of particular note are the illustrations developed for the tutorial, as the Bloch sphere is predominantly a visual tool. It was of critical importance to represent the Bloch sphere and Cartesian axes without significant distortion. For this, a method resembling isometric projection was selected, which offsets all three axes from a straight-on view by a roughly equal amount. This method introduces only minor compromises in perspective that were considered reasonable. Additional care was taken to color-code the Cartesian axes while ensuring that they remained distinguishable even in black and white without becoming too light to see. Efforts were also made to have the diagram read clearly as a sphere at an initial glance, using suggestive shading and guideline cues. These visualizations are supplemented by fully accurate, computer-generated

graphics from a tool independently developed by the University of St. Andrews as part of the QuVis project [136], which is supported by IOP, which students were directed to use to verify their predictions and answers to some questions in the tutorial. With these two types of diagrams working in tandem, it is hoped that students would be able to mentally represent the structure and develop cognitive fluency with its geometry. The tutorial can be found in Appendix E.1.

Developed and validated alongside the tutorial were a pre-test and post-test to evaluate students' understanding of the underlying concepts. The two tests contained isomorphic questions with minor changes to some details such as angles or given states. The post-test versions of these questions are provided in Appendix E.2.

### 6.2.2 Learning objectives

The tutorial consists of three sections, with broad learning objectives as follows:

1. Construction: Students should be able to describe how any single-qubit state can be transformed into the form  $\cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle$ , and depict such a state on the Bloch sphere.
2. Measurements: Students should be able to identify the outcomes of measurements in a given state when a measurement basis is provided, qualitatively describe the probability of obtaining each outcome using visual cues, and explicitly calculate the probability of obtaining a particular outcome when given enough information regarding angles.
3. Geometric intuition: Students should be able to qualitatively describe and diagrammatically indicate answers to various questions, such as the set of all possible states that can yield a particular probability distribution when measured in a given basis.

### 6.2.3 Course implementation

The tutorial was administered at a large research university in the United States, in a multi-disciplinary undergraduate course titled “Foundations of Quantum Computing and Quantum Information.” As the name suggests, the course focuses on an introduction to quantum information and (particularly) quantum computing. Aimed at undergraduate students from all science and engineering majors, it is the only mandatory course for a Quantum Computing and Quantum Information certificate available to interested undergraduate students across disciplines. The class comprised 28 students of diverse backgrounds: seven from engineering science, computer engineering, or industrial engineering; seven from computer science; six from math; two from chemistry; and twelve from physics. Some students chose more than one major from these disciplines. Most of them were sophomores, juniors, and seniors (one first-year student from the College of General Studies was enrolled). The more coarse-grained structure of the course and lack of pre-requisite knowledge related to quantum information enables students of any background to benefit from taking it. The only prerequisites are Calculus I and II, which serve as a proxy to determine students’ ability to engage with the math, predominantly linear algebra, taught in the course in a self-contained way. The curriculum focused on ideas such as two-state systems, quantum gates, and doing measurements, but was less concerned with the details of making a good qubit, correcting errors, and the technical physics behind gating and measurements. Selected as the course text was Thomas Wong’s *Introduction to Classical and Quantum Computing*.

The Bloch sphere tutorial was implemented after traditional lecture-based instruction on the relevant concepts. The students were first given a pre-test to establish a baseline level of knowledge following the lectures and traditional homework, and then they were assigned the

tutorial for homework. Afterwards, once students submitted the homework, they were given the post-test. Common difficulties and performance improvements are discussed in the following sections. Two researchers graded a fifth of the pre- and post-tests. After discussion, they converged on a rubric for which the inter-rater reliability was greater than 90%. Following this, one researcher graded the remaining pre- and post-tests. Each student response to an open-ended question was graded on a three-tiered scale of 0, 0.5, or 1 point, for each salient unit of a problem. (A few problems had multiple aspects that were graded in this way, such as one that asked students to identify two angles.)

### **6.3 Results: Student difficulties**

#### **6.3.1 Overall phase vs. relative phase**

Many students were not especially clear on the identification of quantum states that are equivalent after traditional lecture-based instruction. Equivalent quantum states are described in the tutorial as states that yield the same outcomes in all measurement bases, and in individual interviews, students had no difficulty interpreting the intended meaning of these terms. To evaluate students on their knowledge, question 1 on the pre- and post-test (see Appendix E.2) posed a scenario in which there was a pair of states that differed in overall phase, and one with a pair that differed in relative phase, asking if each pair consisted of equivalent states. On the pre-test, most students answered yes or no for both pairs of states, making no distinction between the two pairs, making the overall correctness rate for the question close to 50% (see Table 6.1). A few stated “no” for the equivalent pair and “yes” for the nonequivalent pair, without indicating particularly

compelling answers for why they provided these responses. Overall, it appears that many students were confused.

The tutorial helps students learn that two states that differ by an overall phase yield outcomes with identical probabilities (in any basis), while two states with different relative phases, which are not equivalent quantum states, may yield outcomes with the same probabilities in one basis but not another. To do this, students were given various pairs of states in the standard basis  $\{|0\rangle, |1\rangle\}$ , and were instructed to calculate the probabilities of measuring each outcome in both the  $\{|0\rangle, |1\rangle\}$  basis and the  $\{|+\rangle, |-\rangle\}$  basis, where  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ .

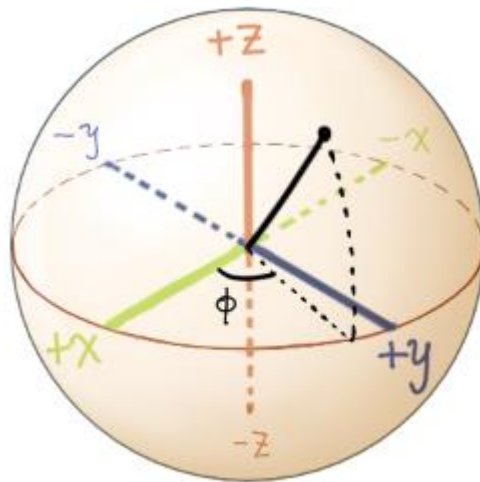
Students performed better on the post-test for this concept, but there is still room for improvement, especially when compared to the other concepts (see Table 6.1). One possible reason is that the cues may have been more subtle for this question than others (e.g., there were no obvious diagrams that could evoke particular concepts from the tutorial). It is also possible that some students were getting too focused on the details of the calculations that were asked of them in the tutorial, and that the cognitive load distracted them from the important conclusion that equivalent states only differ by an overall phase. The illustration used in the tutorial could thus be improved; for example, by providing some of the basis change steps in advance to reduce the effort required to understand the problem statement, or having different hypothetical students make correct and incorrect statements in a written discussion, and asking students to reflect upon the validity of each.

### 6.3.2 Definitions of $\theta$ and $\phi$ on the Bloch sphere

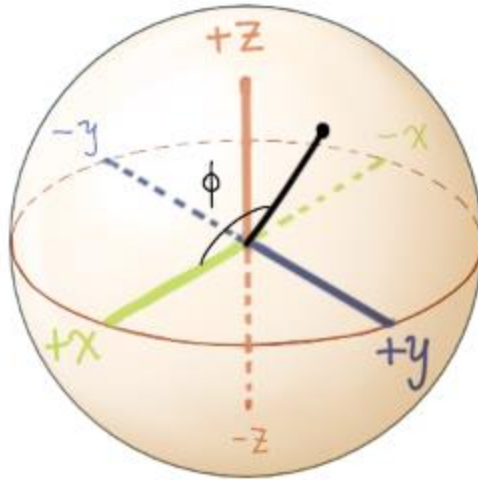
On pretest questions 3a-d, most students correctly associated the angle in the argument of the cosine and sine functions in a state  $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$  with the polar angle. However, they frequently neglected to multiply  $\frac{\theta}{2}$  by 2 to obtain  $\theta$ , resulting in them drawing a polar angle half the required size on the Bloch sphere. The other type of common incorrect answer centered around not starting the angles  $\theta$  and  $\phi$  from their conventional starting points, the  $z$ -axis and  $x$ -axis, respectively. As a further example of this, on a cross-section of the Bloch sphere in questions 5a-b, some students on the pre-test considered the state overlapping with the  $-z$ -axis to be the  $|0\rangle$  state, and the one overlapping with the  $+z$ -axis to be the  $|1\rangle$  state, e.g., because of an association of the “lower” state with a lower energy. This is the reverse of the typical convention, and some of these students’ responses were inconsistent between questions, since they had used the typical convention to answer prior questions.

The tutorial calls explicit attention to the  $\frac{\theta}{2}$  present in the form  $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ , and included an exercise directing students to find the values of the angles for states that corresponded to the Cartesian axes. Both issues discussed above were largely corrected on the post-test, while students were given at least half credit if they explicitly stated the correct values for the angles. It was interesting, however, that on post-test question 3a, some students who explicitly wrote down the correct value of  $\phi$  (which in this case was  $\frac{2\pi}{3}$ ), when drawing the angle on the Bloch sphere, did not extend it past the  $y$ -axis as they should have, affecting their answer to question 3c. This could be due to a careless mistake, but these students who then used their diagram to answer question 3c did so correctly with respect to their diagram.

One last difficulty observed among multiple students, primarily on the post-test because the other difficulties had greatly diminished, was the labeling of not only  $\theta$  (correctly) as the polar angle with respect to the  $+z$ -axis, but also of  $\phi$  (incorrectly) as a polar angle measured with respect to the  $+x$ -axis rather than an azimuthal angle in the  $x$ - $y$  plane. The difference between the two (depicted in Figures 6.1-6.2) is rather subtle, and the tutorial does not go into particularly lengthy detail about the definitions of “polar” and “azimuthal” beyond illustrating the terms with some diagrams.



**Figure 6.1** A conventional depiction in physics of the angle  $\phi$ . This angle is defined as the azimuthal angle in the  $x$ - $y$  plane.



**Figure 6.2** A non-conventional depiction provided by some students. In this depiction, they appear to interpret  $\phi$  as a polar angle with respect to the positive  $x$ -axis (similarly to  $\theta$ , not shown, which is defined with respect to the positive  $z$ -axis).

### 6.3.3 Difficulties with the Born rule

Question 3b on the pre-test and post-test asked students to provide the possible measurement outcomes for the qubit in the standard basis  $\{|0\rangle, |1\rangle\}$ , and their respective probabilities. On the pre-test, students tended to provide dichotomous answers that were either completely correct or not on the right track. The most common difficulty that appeared with some consistency was directly squaring the complex exponential rather than squaring its modulus, though only a few students did this. Only this particular difficulty was present on the post-test as well, and the others had been completely addressed.

### 6.3.4 Measurements made in bases other than the standard basis

The tutorial has a section dealing with measurements made in arbitrary bases, when the polar angle that the state makes with respect to the measurement basis states is known. In this case, if  $\theta$  is the angle between the given state and one of the basis states, the probability that the measurement yields that basis state is  $\cos^2 \frac{\theta}{2}$ . This knowledge was evaluated by pre- and post-test questions 4a-c.

On the pre-test, many students gave multiple incorrect answers. After engaging with the tutorial, the vast majority of students gave answers in the correct form, with some only forgetting to divide the appropriate  $\theta$  by 2, for which they were given partial credit.

### 6.3.5 Identifying states for which a particular measurement outcome is more likely

On questions 5a-c on the pre-test and post-test, students were given a cross-section of the Bloch sphere (i.e., a circle) containing the  $x$ -axis and the  $z$ -axis, corresponding to the  $\{|+\rangle, |-\rangle\}$  and  $\{|0\rangle, |1\rangle\}$  bases, respectively. They were asked to indicate states that had a greater chance of yielding one of the outcomes than the other when measured in each basis. Expected answers are states on the edge of the circle positioned close to, or even directly on top of, the states in question, e.g., a state with a greater chance of yielding  $|0\rangle$  than  $|1\rangle$  when measured in the  $\{|0\rangle, |1\rangle\}$  basis is any state on the upper half of the cross-section shown as a circle.

On the pre-test and the post-test, the most common difficulty for questions 5a-c was for students to indicate states on the interior of the cross-section of the Bloch sphere, rather than the edge of the circle. Such states are interpreted as mixed states, which were beyond the scope of the tutorial and the course, but to the extent that students' indicated states were closer to the correct

basis state, such responses were given partial credit. This difficulty was observed in lower numbers on the post-test, but was not eliminated entirely. One student initially answered with points on the edge of the circle, before erasing these answers and instead choosing points along the axes on the interior of the cross-section. It is possible that at least some students intended such points to represent locations on the surface of the Bloch sphere some distance above or below the page, in which case they would be acceptable answers. However, this remains unclear (these types of answers and reasoning did not come up in interviews before the in-class implementation), and it is also possible that some students had a strong preference for remaining along the axes that represent a measurement basis. They may think, for example, that only states located along the  $\pm z$ -axes may be measured with what they consider “valid” outcomes in the  $\{|0\rangle, |1\rangle\}$  basis.

On the pre-test, it was also relatively common for students to mislocate the  $|0\rangle, |1\rangle, |+\rangle$ , and  $|-\rangle$  states. For some students, this was as simple as switching the  $|0\rangle$  and  $|1\rangle$  states, but others appeared to think that the  $|0\rangle$  and  $|1\rangle$  states were located on the positive and negative  $x$ -axis, in effect rotating the Bloch sphere by a quarter turn counterclockwise. These difficulties appeared to be related only to an initial unfamiliarity with the conventions and definitions of the Bloch sphere, and had disappeared on the post-test.

In the preliminary investigation of difficulties, it was somewhat common for students to explain that the  $|0\rangle$  state lies on the positive  $z$ -axis while the  $|1\rangle$  state lies on the positive  $x$ -axis, which is incorrect. This difficulty likely comes from the states  $|0\rangle$  and  $|1\rangle$  being referred to as “orthogonal” to one another, and such orthogonality of these states does manifest in a right angle between them in two-dimensional Hilbert space. However, in the mapping onto the Bloch sphere, orthogonal states appear diametrically opposite, not at right angles, to each other, and the tutorial emphasizes this point. While in this implementation, only one student on the pre-test provided an

answer to this effect, this difficulty is likely to come up when students receive instruction on the Bloch sphere.

### 6.3.6 Measurements for which the outcome is certain

On questions 6 and 7 on the pre-test, many students did not convey that a qubit would yield an outcome with 100% certainty if and only if the qubit was in one of the measurement basis states. This was true whether they were asked to identify states for which a measurement outcome is certain (question 6), or the number of bases in which a given state could be measured to yield a certain outcome (question 7, in which four possible answers were given: zero, one, two, or infinitely many). On the pre-test, many students left question 6 blank, and all answers in the multiple-choice question 7 were observed; one or two bases were most popular, but at least two students each chose zero or infinitely many bases. (The “zero bases” option was included with the idea that students may mistakenly think that since the state  $|p\rangle$  cannot be measured in the standard basis with 100% certainty, then neither can it be in *any* basis.) This appears to be challenging after traditional lecture-based instruction alone; however, on the post-test after the tutorial, nearly all students answered question 6 correctly, and much improvement was observed on question 7 as well, with three fourths providing the correct response (see Table 6.1).

For question 7, the answer spread was narrowed down to one basis (correct) or two bases. While most students who gave the latter answer did so with no elaboration, a few went on to explicitly give the two bases as ones consisting of the same two states, but written in reverse (e.g.,  $\{|p\rangle, |-p\rangle\}$  and  $\{|-p\rangle, |p\rangle\}$ ), while noting that the two are functionally the same basis. This explanation was given full credit, and it is possible that some of the students who selected two bases as the answer to question 7 had this in mind. However, in interviews, some students were

observed referring to bases by a single state, e.g., “the  $|+\rangle$  basis” or “the  $|p\rangle$  basis,” so it may also be the case that some students did not have yet a fully developed idea of what constitutes a basis.

### 6.3.7 Finding the set of all states given a probability distribution

One other visual affordance offered by the Bloch sphere is the ability to narrow down the possible states a qubit is in after the probability distribution in one or more measurement bases is known. In the case of a qubit that yields  $|0\rangle$  with 60% probability and  $|1\rangle$  with 40% probability, all possible states that have this property are found on a circle on the surface of the Bloch sphere with  $\cos^2 \frac{\theta}{2} = 0.6$ . Visually, this looks like a circle of constant “latitude” located slightly above the “equator” (i.e., the circle that intersects the  $xy$ -plane) of the Bloch sphere. Similar conclusions can be reached given the results of measurements made in any basis.

This was not something that most students were very familiar with immediately following traditional lecture-based instruction, as their answers on question 8 on the pre-test had no consistent underlying patterns. However, it appears that the tutorial was helpful in helping them learn this concept, as evidenced by the exceptional normalized gain and effect size yielding a final performance of 86% on the post-test (see Table 6.1). The tutorial dedicates a section to this visual representation, including questions that made use of the QuVis Bloch sphere visualization [136]. Students were asked to provide qualitative descriptions of what happens to a state on the Bloch sphere while varying either  $\theta$  or  $\phi$  while holding the other constant, and their interaction with this tool along with subsequent questions prodding them draw further conclusions appear to have been effective at helping students learn this concept. Since some students did not appear to be clear about the distinction between an overall and relative phase, there could be a further opportunity to

reinforce the visual meaning of a relative phase with this concept. For example, one could have them contemplate that knowing the measurement probabilities of each outcome fixes an angle  $\theta$  with respect to the measurement basis, but varying the value of  $\phi$  with respect to the basis (while holding  $\theta$  constant) sweeps out a circle on the surface of the Bloch sphere.

A summary of the difficulties discussed in this section can be found in Table 6.2.

**Table 6.1 Comparison of scores before and after the administration of the tutorial, along with normalized gain [120] and effect size as measured by Cohen’s  $d$  [121], for students who engaged with the CQS ( $N = 28$ ).**

	Pre-test mean	Post-test mean	Normalized gain	Effect size
1	66%	73%	0.21	0.11
2	100%	96%	-	-
3a	57%	85%	0.65	0.91
3b	66%	95%	0.84	0.87
3c	93%	96%	0.50	0.16
3d	82%	100%	1.00	0.66
4a	38%	89%	0.83	0.74
4b	36%	82%	0.72	0.64
5a	50%	86%	0.71	0.98
5b	50%	84%	0.68	0.90
5c	66%	84%	0.53	0.52
6	55%	93%	0.84	0.99
7	39%	75%	0.59	0.77
8	29%	86%	0.80	1.55

**Table 6.2 Summary of the difficulties described and student improvement on each between the pre-test and post-test.**

Difficulties	Pre-/post-test #   comments
Difficulties with identifying equivalent states using overall or relative phase	1   Some improvement
Every possible state can be plotted on the Bloch sphere with unique $\theta$ and $\phi$	2   High pre-test performance

Unfamiliarity with Bloch sphere conventions, e.g., which states are defined as $ 0\rangle$ or $ 1\rangle$ ; or the angles $\theta$ and $\phi$ for a given quantum state	3a, 5a   Major improvement
Difficulty identifying possible outcomes and calculating probabilities of those outcomes when measuring a qubit	3b   Major improvement
Determining which outcome is more likely given a state and a measurement basis	3c, 3d   Some improvement
Inability to interpret probability of measuring a particular outcome as a function of the polar angle between the given state and measurement basis state	4a, 4b   Major improvement
Difficulties identifying possible states that are more likely to yield one outcome than another	5a, 5b, 5c   Major improvement, though a number of students tended to plot points on the interior of the Bloch sphere
Not knowing that a quantum state which is a measurement basis state can be measured with 100% certainty	6, 7   Major improvement
Not knowing the set of all states that can be measured with the same probability distribution in a particular basis	8   Major improvement

## 6.4 Discussion

Students who engaged with the tutorial generally performed very well on the post-test. Only two students did not submit a complete tutorial by the deadline, and compared to their peers who did, they exhibited qualitative differences in their responses in addition to performing substantially worse on the post-test. It is possible that these differences are due to the fact that those two students did not engage with the Bloch sphere tutorial.

Physics majors and non-physics majors did not display appreciable differences in performance on the pre- and post-test. This may not be especially surprising, as the Bloch sphere is not typically discussed in detail in QM courses in a typical physics program.

However, especially on the pre-test, the geometry of spherical coordinates proved difficult for many students, which may be because spherical coordinates are not broadly covered in different science and engineering disciplines. That said, the tutorial greatly increased students' facility with spherical coordinate notations and conventions, as seen in improvement in questions 3a-d (see Table 6.1). In this, physics majors did appear to be more comfortable than non-physics majors, which suggests that their previous physics courses had introduced them to spherical coordinates.

Overall, findings from the pre- and post-tests suggest that the only concept that may need some refinement is the one evaluated by question 1, which asks about equivalent quantum states. (Question 7 has a similar post-test score, but much higher normalized gain and effect size from the lower pre-test score—see Table 6.1.) We are considering approaches to help students engage with this concept more succinctly and reduce their cognitive overload, as well as to encourage students to make connections back to this concept at several different relevant points in the tutorial.

## **6.5 Ethical statement**

This research was carried out in accordance with the principles outlined in the University of Pittsburgh Institutional Review Board (IRB) ethical policy, the Declaration of Helsinki, and local statutory requirements. Informed consent was obtained from all interviewed students who participated in this investigation.

## **6.6 Acknowledgments**

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## **7.0 Investigating and improving student understanding of the basics of quantum computing**

### **7.1 Introduction**

Quantum information science and engineering (QISE) is an exciting interdisciplinary field that has applications in quantum computing, quantum communication and networking, and quantum sensing, which are attractive to scientists and engineers for many reasons. Computer scientists and engineers are developing quantum algorithms for various problems, including ones that become impractical for classical computers to solve at large scales. For example, on a classical computer, the problem of factoring products of large prime numbers scales exponentially with the size of the prime numbers, but on a quantum computer utilizing Shor's algorithm, the problem scales roughly as a polynomial instead. Given that the difficulty of factoring products of large prime numbers is at the heart of the RSA (Rivest-Shamir-Adleman) protocol used for encryption of sensitive data, the development of Shor's algorithm showing a quantum computer's exponential speed-up provided a major impetus to QISE research. As another example, Grover's algorithm also provides an advantage for quantum computers over classical computers in searching an unsorted list, this time as a quadratic speed-up. Since this type of list is often encountered in many applications, even a quadratic speed-up constitutes a major improvement. For future applications in science, physicists and chemists are also excited about the potential of quantum computers to solve important problems in their disciplines in which solving the Schrödinger equation plays an important role. The development of robust quantum bits (qubits) and scalable quantum computers demands the expertise of physicists and engineers alike. For all these reasons and more, this area

of study holds a great amount of promise for students from many science and engineering disciplines interested in careers in QISE-related fields [27,30,28,29,31,32].

To ensure their success in the field, students must develop a strong foundational knowledge of qubits and quantum systems at the start of their studies, as well as a sense of the advantages that quantum computers are capable of providing over classical computers. Here we describe the development, validation and in-class implementation of a research-based Quantum Interactive Learning Tutorial (QuILT) on these topics. The tutorial was implemented in two types of courses at a large research university in the United States: one is the standard two-semester quantum mechanics (QM) course sequence for physics majors, and the other is an interdisciplinary course involving students from many fields, the only mandatory course in a new “Foundations of Quantum Computing and Quantum Information” undergraduate certificate program offered by the institution to undergraduate students across science and engineering disciplines.

Learning QM is challenging for students partly since the quantum paradigm is very different from the classical paradigm [1]. Prior research suggests that students in QM courses often share common difficulties, but that research-validated learning tools can effectively help students develop a functional understanding [1–26,139]. We have been researching student difficulties after traditional lecture-based instruction and using the research to guide the development of learning tools for concepts covered in undergraduate QM courses. Previous work from our group includes QuILTs on topics such as the Mach-Zehnder interferometer and quantum key distribution, and Clicker Question Sequences on topics such as the basics and change of basis, quantum measurement, time-development, and measurement uncertainty of two-state quantum systems [51,75,115,122,137,138]. We find that after traditional lecture-based instruction, students may not have a strong grasp of important quantum concepts, but after further engagement, e.g., with a

research-based QuILT, they may develop additional fluency with those concepts. To that end, we have developed and validated a QuILT (referred to in this chapter as “the tutorial”) to help students learn about the basics of quantum computing.

Given the interdisciplinary nature of the QISE field, for which this tutorial is intended, there is a need to standardize the language to be accessible and unambiguous for everyone regardless of background. Some trends have already taken hold in the field, including a distinction between interpreting phenomena “classically” as opposed to “quantumly,” rather than “quantum mechanically”; speaking of “measuring qubits” as opposed to measuring a physical observable; referring to a “measurement basis”; or measuring specific states such as  $|0\rangle$  or  $|1\rangle$  as outcomes rather than the corresponding observables (eigenvalues). These are linguistic constructions that quantum physicists have typically not used in the past. However, as there is a one-to-one correspondence, e.g., between the eigenvalues obtained by making a measurement of an observable in a specific state, and the eigenstates that represent the standard basis as  $|0\rangle$  and  $|1\rangle$ , it can serve as useful shorthand to say that a qubit is measured to be in the  $|1\rangle$  state (or measured to be 1) to convey that a measurement made on the system yields the eigenvalue corresponding to the  $|1\rangle$  state.

In this research on the development, validation, and in-class implementation of the quantum computing tutorial, we opt to use the prevailing terminology in the QISE field so that students can become familiar with the language used by their textbooks, instructors, and other professionals in their studies in this interdisciplinary field. Not coincidentally, this language also serves to express many concepts more directly and in fewer words, and thus may reduce the cognitive load to learn them, as many of the physical details are immaterial to many of the QISE contexts in which these students will apply the concepts that they learn. All that said, it is still

crucial to maintain the integrity of language used to educate physicists involved in QISE who will be working to develop robust qubits and build real quantum computers, work that very much requires understanding the entire quantum physics taught in typical undergraduate and graduate physics courses.

## **7.2 Theoretical framework**

In QM courses, whose content can be difficult for students, it is critical to consider constructivist research-based pedagogical approaches to engage students and help them learn these challenging, foundational concepts [1]. The theoretical framework that guided the development, validation and evaluation of this tutorial emphasizes that the research-based pedagogical approaches should balance efficiency and innovation, i.e., Schwartz et al.'s Preparation for Future Learning (PFL) framework [85]. In one interpretation of this framework, innovation in instructional approaches would focus on providing students opportunities to engage and struggle with novel problem-solving tasks so that they develop high order thinking skills and the ability to apply existing knowledge to novel situations (transfer of knowledge). Exploratory labs are examples of environments that are intended to maximize innovation. On the other hand, efficiency in instructional approaches can refer to efficient ways to communicate relevant knowledge to students, e.g., via lectures. The authors observe that, in most traditional classrooms, efficiency is overemphasized while innovation is typically disregarded, and suggest that balancing innovation and efficiency in learning activities can improve conceptual understanding and transfer of knowledge to new contexts [85].

The advantages of balancing innovation and efficiency are outlined by Nokes-Malach and Mestre [89], who propose mechanisms for the phenomenon of the “time for telling” in which students who have productively struggled with innovative invention tasks learn more deeply from subsequent, more efficient methods of instruction than students who did not undergo the invention tasks [116]. They hypothesized that allowing students to productively struggle through invention tasks imparts a “hidden” effectiveness in key ways. For example, the particular framing and productive struggle may prime students to operate in a more mastery-based orientation rather than a performance-based one; a mastery orientation is one in which students are interested in deeply understanding the material, while performance orientation implies students’ desire to, e.g., get a good score or pass the course [117]. On the other hand, the students who only receive lecture-based instruction (an instantiation of efficiency) may use problem-solving steps via a template without comprehensively considering the reasoning for each step [89]. Thus, balancing innovation and efficiency is vital for priming all students to engage in deep sense making and learning within the time-constraints of the course.

Having outlined the benefits of balancing innovation and efficiency, the aforementioned fixation on efficiency in most solely lecture-based classrooms leads to what Schwartz et al. term “routine experts,” who are fluent in a specific type of task but struggle with transferring their learning to new situations, rather than “adaptive experts” who are able to complete a wide variety of tasks requiring transfer of knowledge in their area of expertise with speed and accuracy. A pure focus on innovation leads to another issue: the “frustrated novice,” who without sufficient guidance and scaffolding support, is unable to make any meaningful progress on problems in a given amount of time, ultimately experiencing few benefits of the freedom provided by innovation and discovery. Thus, Schwartz et al. conclude, developing students’ expertise in a domain requires

balancing of efficiency and innovation axes in the instructional design, outlining an “optimal adaptability corridor” by which they can develop competencies to become “adaptive experts” with the least amount of wasted effort [85].

Inspired by this framework, the tutorial on the basics of quantum computing strives to balance innovation and efficiency using guided inquiry-based teaching-learning sequences that build on one another as well as on student prior knowledge to help them develop a functional understanding. The development and validation of our tutorial involves conducting cognitive task analysis from both expert and student perspectives. From the expert perspective, we delineate a fine-grained scope of content and its relevance to our learning objectives through discussions and the iteration of ideas among ourselves many times, as well as consulting multiple faculty members experienced in teaching QM and related fields (e.g., solid state physics). The cognitive task analysis from the student perspective involves interviewing students so that student difficulties can be used as a guide and the teaching-learning sequences in the tutorial provide appropriate scaffolding at a level that balances efficiency and innovation. The cognitive task analysis from the student perspective is valuable to minimize expert blind spots.

## **7.3 Methods**

### **7.3.1 Development and validation**

To use common student difficulties with underlying concepts as a guide and balance innovation and efficiency in the tutorial, student difficulties were investigated over many years, both formally and informally, including via responses to open-ended questions asked on exams in

an undergraduate QM course following traditional lecture-based instruction on the requisite concepts. These difficulties were then used as a guide over the subsequent development and validation of the tutorial. Four students were interviewed several times using a think-aloud protocol, spanning a total of roughly fifteen hours. Their feedback was used to gauge overall flow and whether the tutorial was at the appropriate level, as well as to identify blind spots, and their suggestions were incorporated into the subsequent versions of the tutorial. Throughout, the tutorial was repeatedly iterated with constant discussions among the authors, and with four additional faculty members experienced in teaching QM and solid-state physics continually contributing feedback as well. The tutorial is included in Appendix F.1 and the Supplemental Material for convenience. Developed and validated alongside the tutorial were a pre-test and post-test to evaluate students' understanding of the underlying concepts. The two tests contained isomorphic questions with minor changes to some details such as given states. The post-test versions of these questions are provided in Appendix F.2.

### **7.3.2 Learning objectives**

The learning objectives for the tutorial are such that students should be able to do the following:

- Identify possible states for 2-bit/qubit systems
- Identify examples of what can be used as a bit or qubit
- Identify possible outcomes of measurement as well as calculate probabilities of measuring each outcome in a given state
- Identify whether there are  $N$  or  $2^N$  of a given quantity involved in classical vs. quantum computers (e.g., describe that measurement collapses the state so that only  $N$  bits of

information are obtained as output of computation even for an  $N$ -qubit quantum computer; only  $N$  qubits must be initialized in a quantum computer; a quantum state can in general evolve as a superposition of  $2^N$  states but a classical computer which also has  $2^N$  distinct states can only be in one of those states at a given time, etc.)

- Define and identify examples of superposition states and entangled states and be able to contrast the phenomena of superposition and entanglement
- Describe the role and action of a single-qubit quantum gate (describe how these gates can be used to change a state)
- Specify the bra state corresponding to a given ket state (e.g.,  $|\chi\rangle = a|0\rangle + b|1\rangle$  where  $a$  and  $b$  are in general complex)
- Calculate an outer product and be able to apply an outer product (operator) to a given state to find the outcome
- Write a single-qubit gate in Dirac notation

### 7.3.3 Course implementation

The tutorial was administered at a large research university in the United States in two separate courses. The first was the standard two-semester QM course for junior-/senior-level physics majors, for which data were collected from two different implementations of the course in successive years by different instructors. Both instructors are physics education research-friendly and have used research-based learning tools before. There were 13 students in the first year and 22 students in the second year, for a total of 25 students. The tutorial was given by both instructors after the students had already learned the relevant foundational quantum mechanical concepts via

traditional lecture-based instruction. The course text for this junior/senior level course was McIntyre's *Quantum Mechanics: A Paradigms Approach* in one year and Griffiths' *Introduction to Quantum Mechanics* in the other year, with additional material related to QISE covered in lecture by both instructors.

The second course was a multi-disciplinary undergraduate course titled "Foundations of Quantum Computing and Quantum Information." As the name suggests, the course focuses on an introduction to quantum information and (particularly) quantum computing. Aimed at undergraduate students from all science and engineering majors, it is the only mandatory course for a Quantum Computing and Quantum Information certificate available to interested undergraduate students across disciplines. The class comprised 28 students of diverse backgrounds: seven from engineering (engineering science, computer engineering, or industrial engineering); seven from computer science; six from math; two from chemistry; and twelve from physics. Some students chose more than one major (double major) from these disciplines. Most of them were sophomores, juniors, and seniors (one first-year student from the College of General Studies was enrolled). The structure of this particular course enables students of any background to benefit from taking it. The only prerequisites are Calculus I and II, which serve as a proxy to determine students' ability to engage with the math, predominantly linear algebra, taught in the course in a self-contained way. The curriculum is generally focused on ideas such as two-state systems, quantum gates, and doing measurements, but is less concerned with the details of making a good qubit, correcting errors, and the technical physics behind gating and measurements. Selected as the course text was Wong's *Introduction to Classical and Quantum Computing*.

In total, the data that are matched for the same students across the pre-test and post-test comprise 18 students in the QCQI course (hereby referred to as "QCQI students") and 35 students

in the physics course (referred to as “Physics students”) who completed the pre-test, tutorial, and post-test. The 18 QCQI students who engaged with the tutorial were a subset of the 28 students in the QCQI course mentioned earlier. Additional unmatched data exist for the remaining 10 students who took the post-test but not the pre-test. These data are presented in Appendix F.3 and discussed separately. Also in Appendix F.3 are the results for Physics students differentiated by class, though in the main text we describe the results for the two Physics classes in aggregate.

The quantum computing tutorial was implemented as homework after students had traditional lecture-based instruction on the relevant concepts. The students were first given the pre-test to establish a baseline level of knowledge following the lectures and traditional homework, and then they were assigned the tutorial for homework. Afterwards, once students submitted the homework, they were given the post-test. For student grade, the pre-test counted for completeness grade and the post-test counted for correctness grade for their quizzes. Common difficulties and performance improvements are discussed in the following sections. Because some of these questions were discovered during student interviews to be relatively easy for Physics students, one of the Physics instructors opted not to include these questions (marked in this text with an asterisk [\*]) over time and length concerns. Two researchers graded a fifth of the pre- and post-tests. After discussion, they converged on a rubric for which the inter-rater reliability was greater than 90%. Following this, one researcher graded the remaining pre- and post-tests. Each student response to an open-ended question was graded on a three-tiered scale of 0, 0.5, or 1 point, for each salient unit of a problem. This was the case for all but three questions, which were scored all-or-nothing out of 1 point.

## 7.4 Results

Overall, both Physics students (those in the QM course for physics majors) and QCQI students (those in the Foundations of Quantum Computing and Quantum Information course) did very well on the post-test after engaging with the tutorial. In both courses, only a handful of questions remained somewhat difficult for students after the tutorial, and there were many for which they performed well even after lecture-based instruction. The data are shown in Tables 7.1-7.2. The post-test questions are reproduced in Appendix F.2. For additional insight discussed in later sections, Appendix F.3 presents data for the QCQI students' performance split into physics majors and non-physics majors (Appendix Tables 3-4).

**Table 7.1 Results for Physics students ( $N = 35$ ; questions with an asterisk [\*] have  $N = 22$ ). Pre-test and post-test average scores, normalized gain [120], and effect size as measured by Cohen's  $d$  [121] are presented.**

**Student data for these four columns are matched.**

Question	Pre	Post	Norm. Gain	Effect size
1a	99%	100%	1.00	0.24
1b	99%	100%	1.00	0.24
1c	63%	91%	0.77	0.91
1d	86%	92%	0.45	0.33
2a	100%	99%	--	--
2b	33%	50%	0.26	0.59
3a	91%	100%	1.00	0.43
3b	89%	96%	0.63	0.29
3c	73%	91%	0.68	0.55
3d	46%	69%	0.42	0.69
4a	30%	81%	0.73	1.32
4b	54%	97%	0.94	1.34
4c	53%	89%	0.76	1.17
4d	56%	84%	0.65	0.70
5	43%	87%	0.78	1.08
6	26%	80%	0.73	1.30
7a	80%	97%	0.86	0.61
7b	53%	77%	0.52	0.62
7c	87%	100%	1.00	0.56

8a	70%	91%	0.71	0.64
8b	71%	96%	0.85	0.75
8c	83%	97%	0.83	0.60
8d	61%	87%	0.67	0.67
*9a	82%	100%	1.00	0.72
*9b	82%	100%	1.00	0.72
*9c	91%	95%	0.50	0.20
*10a	91%	95%	0.50	0.27
*10b	70%	77%	0.23	0.21
*10c	66%	89%	0.67	0.70
*10d	57%	89%	0.74	0.82
*11	30%	73%	0.61	1.03

**Table 7.2 Results for QCQI students ( $N = 18$ ). Pre-test and post-test average scores, normalized gain, and effect size as measured by Cohen's  $d$  are presented. Student data for the first four columns are matched, with an additional column containing (unmatched) data from all students who completed the post-test ( $N = 28$ ).**

**(Question 10d was not asked of the QCQI students.)**

Question	Pre ( $N=18$ )	Post ( $N=18$ )	Norm. Gain	Effect size	Post ( $N=28$ )
1a	100%	97%	--	--	98%
1b	100%	96%	--	--	97%
1c	58%	89%	0.73	0.63	82%
1d	86%	93%	0.50	0.21	93%
2a	100%	100%	--	--	100%
2b	50%	67%	0.33	0.35	55%
3a	100%	94%	--	--	93%
3b	86%	100%	1.00	0.34	96%
3c	78%	94%	0.75	0.39	93%
3d	47%	69%	0.42	0.49	64%
4a	50%	86%	0.72	0.70	89%
4b	61%	89%	0.71	0.55	89%
4c	56%	92%	0.81	0.76	87%
4d	67%	92%	0.75	0.51	86%
5	50%	81%	0.61	0.53	86%
6	39%	83%	0.73	0.78	82%
7a	78%	94%	0.75	0.34	95%
7b	47%	78%	0.58	0.58	75%
7c	69%	100%	1.00	0.64	100%

8a	86%	92%	0.40	0.13	95%
8b	89%	100%	1.00	0.30	98%
8c	81%	90%	0.50	0.24	90%
8d	58%	89%	0.73	0.57	82%
9a	89%	100%	1.00	0.28	96%
9b	89%	100%	1.00	0.28	100%
9c	92%	94%	0.33	0.07	96%
10a	72%	100%	1.00	0.66	98%
10b	61%	67%	0.14	0.11	71%
10c	47%	74%	0.50	0.52	67%
11	25%	64%	0.52	0.70	46%

## 7.4.1 Difficulties assessed for all students

### 7.4.1.1 Questions with performance $\geq 85\%$ after lecture-based instruction

Questions 1a and 1b asked students to identify all the possible independent states for two-bit and two-qubit systems. They overall did very well. For classical systems, the four states are the only ones available, while quantum systems are allowed to be in any linear superposition of the analogous four linearly independent basis states. Only in rare instances did students answer that both bits or qubits must match in the possible states of the system, thus yielding two possible states for their final answer. Out of all the students, this happened only for one Physics student on the pre-test and one QCQI student on the post-test. Another QCQI student stated that infinitely many states are available, such as  $\{|00\rangle, |01\rangle, |+\ -\rangle, |+\ +\rangle, \dots\}$ , so long as the states within the ket brackets are orthonormal. This was most likely connected to the fact that there are infinitely many bases that can be chosen for a quantum system. While this is true, the states are not all independent from one another, and so they cannot all be used as part of the response to the question. Interestingly, both QCQI students had answered the question correctly on the pre-test (see Tables

7.1-7.2). Though exhibited rarely in these data, these are likely to be difficulties characteristic of students who may struggle with these concepts.

Additionally, students did well with Questions 2a and 3a. In each case, they were asked to provide the probabilities of obtaining certain outcomes when a qubit in a given state is measured (a specific arbitrary state for Question 2a, and the state  $|-\rangle$  for Question 3a when measured in the  $\{|+\rangle, |-\rangle\}$  basis). Students in both courses scored nearly perfectly on both questions, indicating that lecture-based instruction was sufficient for these concepts.

Students also did reasonably well on Question 3b, which dealt with consecutive measurements made in a different basis; any mistakes made on the pre-test were minor and were overcome on the post-test (see Tables 7.1-7.2).

#### **7.4.1.2 Physical things that can be used as bits or qubits**

Question 1c asked students to give examples of physical entities that could be used to encode classical bits and qubits. Students in both courses improved on this question, mostly in their given examples of qubits. On the pre-test, some students did not provide answers for one or both parts of the question; of those who did, many students were able to give adequate examples of classical bits, though some responses like “a circuit,” “an AND gate,” or “voltage (on a wire)” were too vague and did not include the crucial element (e.g., an on-off switch for the circuit) that would have given their example two discernible states. On the pre-test, examples for qubits were somewhat less clear, including an issue of a similar nature (e.g., invoking electrons or photons very generally without specifying the two basis states that would make them suitable qubits; some students explicitly referred to the position of an electron). In addition to the preceding answers, some others appeared to refer to quantum systems for both answers, such as giving “ground and excited [states of a] hydrogen atom” or “a particle trapped in a box” as a classical bit.

In one or two cases on both the pre-test and post-test, students clearly described an idea that a classical bit is digital while a qubit is analogue, such as saying a pipe with water flowing or not can be taken as a classical bit, while the amount of water in the pipe would constitute a qubit. Another student gave the example of a switch (classical) and a slider (quantum). This may be an attempt to incorporate the knowledge that a qubit can be in any linear superposition of two basis states, in some sense being “between” the two states, but it does not take the quantum behavior of the system, including measurement collapse, into account. One of the experts consulted during the development of the tutorial had characterized a qubit using similar language with regard to this “analogue” nature, so depending on the context, this can be a useful way of describing the concept. Some students specifically drew the distinction that a qubit can be in a superposition or “combination of states,” while a classical bit could only be in one state, which is a helpful direction even if they were not specific about the type of superposition.

A small number of students responded by saying that the same thing, such as the heads and tails sides of a coin, can be used for both. Some students were on the right track but gave somewhat incomplete answers for a qubit, such as “an electron” or “particle spin” (because only spin-1/2 particles such as electrons would constitute a two-state system). Some students gave somewhat circular answers, such as saying “a bit string” when asked for a classical bit, or restating the fact that classical computers use classical bits and quantum computers use quantum bits. Responses of all these types were found on both the pre-test and the post-test, but more students gave correct and much more uniform answers on the post-test (see Tables 7.1-7.2).

The rate of correct answers climbed from around 60% to ~90% on the post-test (see Tables 7.1-7.2). The vast majority of students suitably indicated the nature of their provided examples of both classical bits and qubits on the post-test.

### 7.4.1.3 Probabilities of measuring each possible outcome

Question 1d gave students a generic state  $a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ , asking them what the probabilities are of measuring each of the four possible outcomes ( $|a_{ij}|^2$  for  $i, j = 0, 1$ ) and for the sum of these probabilities ( $\sum |a_{ij}|^2 = 1$ ). Since most students indicated that the probabilities should sum to 1, they were awarded a baseline of half credit for this question regardless of their answer to the other part. However, on the pre-test, many students said that a measurement would result in any of the four outcomes each with a  $\frac{1}{4}$  probability, without obvious justification for why this should be so. Others, particularly the Physics students, used the  $a_{ij}$  coefficients, but took the simple square instead of the square of the modulus. This may be a difference in what the courses emphasized; the physics students could have seen a preponderance of real-number-only examples in class examples and homework assignments, while the QCQI class may have used many complex numbers, kept things abstract, or constantly reinforced the importance of the absolute value brackets, etc. Starting from relatively high pre-test scores, students in both courses improved a bit more on the post-test (see Tables 7.1-7.2).

### 7.4.1.4 Application of Hadamard gate to a state

Question 3c asked students for the result of applying a Hadamard gate to the  $|1\rangle$  state, and the possible outcomes and their associated measurement probabilities. Most students got this question fully correct on the pre-test, but the ones who didn't left it blank, answered only one of the two parts correctly, made little progress, or wrote the Hadamard gate correctly but mistakenly wrote the  $|0\rangle$  state (as they were asked on the pre-test) as  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  instead of  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , which made them conclude that application of the gate to the null vector results in the null vector. This misinterpretation of the  $|0\rangle$  as the null vector was observed in only one of the two groups of

Physics students. Post-test performance on this question is high (see Tables 7.1-7.2), though students were asked to apply the Hadamard gate to the state  $|1\rangle$  instead of  $|0\rangle$ , and so they were not given the opportunity to demonstrate whether they had corrected this particular error.

#### 7.4.1.5 Measurement of the output of a quantum computer

Related to the preceding discussions of measuring outcomes in Questions 1d, 2a, and 3a-c, in which students did reasonably well on both the pre-test and post-test, Questions 7a and 7c apply these ideas to a quantum computation. The trend continues, with students achieving comparatively high scores on the pre-test after lecture-based instruction, and further improvements on the post-test. The questions ask what the probability is of measuring the outcome associated with  $|01\rangle$ , assumed to be the “correct answer” within the context of the problem, with the given state being  $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$  in 7a and  $\sqrt{\frac{1}{300}}|00\rangle + \sqrt{\frac{99}{100}}|01\rangle + \sqrt{\frac{1}{300}}|10\rangle + \sqrt{\frac{1}{300}}|11\rangle$  in 7c.

Question 7b gave the generic state  $a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$  and asked students what  $a_{ij}$  must be for a measurement to output the correct answer  $|01\rangle$ . On the pre-test, a large number of students stated clearly that  $a_{01}$  must be 1 while the remaining  $a_{ij}$  must be 0. Other students, particularly in the Physics course, were not as clear in their answers, e.g., saying  $a_{ij} = 1$  without specifying any particular  $i$  or  $j$ ; still other students in both courses appeared completely lost or left the question blank. Students were still awarded partial credit if they said that the desired state can be achieved by applying the correct quantum gates. Question 7b is further discussed later in connection with other questions involving quantum gates. Students did much better on the post-test, with scores rising to 75-80% (see Tables 7.1-7.2).

#### 7.4.1.6 $N$ entities vs. $2^N$ entities

Before engaging with the tutorial, after traditional lecture-based instruction, one reasoning primitive [140] that was common in student responses is that a major difference between an  $N$ -bit classical and  $N$ -qubit quantum computer is that various things that are  $N$  for a classical computer should be replaced with  $2^N$  for a quantum computer (e.g.,  $2^N$  qubits must be initialized and  $2^N$  bits of information are obtained as the output of the computation on the quantum computer). This type of reasoning primitive also led many students to incorrectly think that there are only  $N$  distinctly different states available when computation takes place on a classical computer. It also led many students to use as a heuristic the notion that the processing and storage advantage of quantum computers can be understood by replacing  $N$  for a classical computer with  $2^N$  for a quantum computer. Research suggests that this type of reasoning primitive has its origins in students learning that quantum computers can provide exponential advantage for certain problems, e.g., Shor's algorithm for factoring products of large prime numbers, and that the quantum state during the computation can be in a superposition of  $2^N$  linearly independent states.

Even in our earlier investigations before the development of the tutorial, we found that students often had difficulty correctly ascribing the numbers  $N$  and  $2^N$  to various quantities for a classical and quantum computer with  $N$  bits or qubits, respectively. Therefore, across the pre-test and post-test that accompany the tutorial, this distinction between the two numbers ( $N$  vs.  $2^N$ ) was asked across several questions on the pre-test and post-test in multiple forms: in terms of number of operations during initialization (Questions 4a and 6), number of bits of information extractable from the output (Question 4b), number of linearly independent parameters necessary to describe a quantum computer and whether this severely limits its size (Question 4c), and number of available states or basis states (Question 5). For all these questions except 4c, the cases of classical and

quantum computers were juxtaposed with one another; Question 4c contained this juxtaposition implicitly, since obviously classical computers with  $N > 300$  can be built with no issue. The core issue is the tendency of many students to think that an  $N$ -qubit quantum computer has more linearly independent states than a  $N$ -bit classical computer, likely the result of the reasoning primitive suggesting that  $N$ -qubit quantum computers are associated with  $2^N$  qubits or states while classical computers are associated with  $N$  bits or states.

Pre-test performance on Questions 4a-c, all true/false questions with explanation, was low among the Physics students and somewhat higher among QCQI students, but improved substantially on the post-test in both courses. In each case, students were observed to defend their incorrect answers, provide no explanation, or leave the questions blank entirely on the pre-test. On the post-test, they tended to answer with more clarity and correct explanations.

Question 4a was particularly difficult for students in both courses, with many students on the pre-test agreeing with the statement that quantum computers must initialize  $2^N$  qubits while classical computers need to initialize only  $N$  bits. This is consistent with their answers in Question 6, for which the correct number of operations that must be performed to initialize an  $N$ -qubit quantum computer is  $N$ ; many students chose  $2^N$  on the pre-test. On the post-test, students performed better on both questions (see Tables 7.1-7.2), frequently converging on the reasoning that a classical and quantum computer of equal size must both initialize the same number of qubits. Most of those students identified that number to be  $N$ , though a small number said  $2^N$  instead. This was observed on both the pre-test and the post-test. As these two questions, which did not appear consecutively, were effectively asking the same thing, students' consistency in their answers is a good sign that they are understanding and reasoning through what they are being asked.

For Question 4b, on the pre-test students did not agree that a quantum computer's output, the result of a quantum measurement, contains an amount of information equivalent to that of a classical  $N$ -bit string. Again, many students gravitated toward the idea that a quantum computer must in some way contend with more information than an equivalently sized classical computer. This seems a natural assumption if one is to think at face value that quantum computers can offer some advantage over classical computers. After engaging with the tutorial, most students on the post-test correctly noted that both classical and quantum computers' outputs contain the same amount of information (see Tables 7.1-7.2).

It appears that students develop the sense that an  $N$ -qubit quantum computer has more linearly independent states than an  $N$ -bit classical computer has states (both have  $2^N$ ) because a quantum computer can evolve its qubits in various states of superposition or entanglement during a quantum computation. In this vein, Question 4c appeals to the notion that keeping track of the amplitudes of each of these basis states, a problem not encountered on classical computers, is an infeasible task as  $N$  becomes large. On the pre-test, many students agreed with the incorrect statement that this issue prevents large quantum computers from being built. Encouragingly, on the post-test, most students provided valid answers (see Tables 7.1-7.2), often stating explicitly that a quantum computer "keeps track" of its own qubit states and does not need to store associated information in classical registers, thus acknowledging that operating a quantum computer is different from simulating a quantum computer on classical hardware.

Students also learn that an  $N$ -qubit quantum computer can be in a superposition of all  $2^N$  basis states while an  $N$ -bit classical computer can only be in one of the possible states at any time. It may not be explicit to students that an  $N$ -bit classical computer has  $2^N$  possible states, and there is often a temptation to assert that there are fewer. This is consistent with the reasoning primitive

that a classical computer ought to have  $N$  available states. We observe on the pre-test that very many students in both courses agreed with the student in Question 5 who says, incorrectly, that a classical computer has only  $N$  states available to it. Some students defended Student 1's reasoning while others provided no further explanation. The correctness figure of this question rose to 80% on the post-test (see Tables 7.1-7.2), with the vast majority of students correctly agreeing with Student 2, many again citing that the number of possible classical states and quantum basis states should be equal.

#### **7.4.1.7 Superposition vs. entanglement**

Question 4d investigates whether students can distinguish between superposition and entanglement. It is not uncommon for students to confuse the two, even though they describe distinct phenomena. Entanglement becomes salient when there are two or more qubits involved in a system of interest, but many students easily make the overgeneralization that any multi-qubit state that is a linear combination (i.e., superposition) of the possible basis states is an entangled state. Only multi-qubit states that cannot be written as a product of states of each qubit are entangled. For example, entanglement can make the outcomes of measurement of the qubits dependent on each other. To entangle qubits, the qubits in question must have interacted either directly or indirectly with one another beforehand, a concept that was challenging for many students.

The pre-test scores were low for the Physics students and higher for the QCQI students. In both courses, students' incorrect pre-test responses tended to be ones without elaboration or blank, suggesting they were not quite sure of their answers. Those QCQI students who answered the question correctly on the pre-test supplied the reason that only states that are non-factorable (in any basis) as the product of states for each qubit are considered entangled, and that not every

superposition state of multiple qubits fulfills this criterion. While there are many common valid justifications for disagreeing with the incorrect statement, this was the most common one, as it was likely to be the one most emphasized in class. The tutorial helped students reflect upon this type of explanation as well as others. Both Physics and QCQI students improved on the post-test, and some also noted that, for a multi-qubit system, the set of all superposition states is larger than the set of all entangled states.

Questions 8a-d also probe students' knowledge of these concepts. Question 8a asks students if a single qubit can be in a superposition state, Question 8b asks if a single qubit can be in an entangled state, and Question 8c asks if multiple qubits can be in an entangled state. The QCQI students did quite well on the pre-test for Questions 8a-c, while the Physics students had somewhat lower performance. For example, a small number of Physics students (but no QCQI students) were prone to saying that a single qubit can be entangled, usually with no elaboration; one student took for granted that entanglement was possible, saying only that entanglement is present for some but not all states, which is true only for multi-qubit systems. The number of affirmative responses for 8b decreased on the post-test. In both courses, pre-test performance on Question 8d, which specifically asked if qubits can become entangled without any prior interaction, was lower than for 8a-c. Question 8d was intended to be slightly trickier than the rest, and a number of students stated that entangled qubits need not have prior interactions (direct or indirect), with some saying that simply passing them through an appropriate gate, or that preparing them together at the same time, would entangle them. These students appear to have learned about gates, e.g., control not (CNOT) that can entangle qubits, but had not yet made the connection that a gate like CNOT that entangles qubits must involve some sort of direct or indirect qubit interaction. It may point to a rather mechanical approach to learning these types of concepts,

invoking specific gates without consideration, e.g., to what applying those gates to a multi-qubit system would physically imply. Also on the pre-test, a number of students did not provide elaboration, pointing to the likelihood that they were unsure of the reasoning behind their affirmative or negative answers. For all of these questions, all students in both courses did very well on the post-test (see Tables 7.1-7.2).

#### **7.4.1.8 State of student knowledge on single-qubit quantum gates**

Questions 2b asked students how a single-qubit quantum gate might be constructed to transform an arbitrary given state to one of the basis states. Similarly, Question 3d asked students to provide a quantum gate that would transform one basis state to the other basis state (for which the Pauli X gate  $\sigma_x$  is an example). Since numerically solving for such a quantum gate can be quite mathematically involved, we considered qualitative responses specifying some sort of rotation of the input state perfectly acceptable. Students typically either gave this type of response associating the application of quantum gates to rotation of the state, or alternatively provided a gate represented by a matrix for such questions, which almost always were projective measurement gates such as  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  or one of the familiar Pauli gates. A few students even explicitly mentioned that the gate should collapse the given state to the desired outcome, not recognizing the significance of the unitarity of quantum gates. Some gave gates like NOT or CNOT which were either classical or multi-qubit in nature. Other students simply responded to Question 3d by affirming that such a gate does exist, without providing further elaboration or the gate itself.

It was interesting that, for both questions, there was a distinct split between courses in how students tended to answer this question. Only about half of QCQI students, but nearly all Physics students, gave a matrix as an answer, and some students in both courses even acknowledged that

their chosen matrix could not be a valid quantum gate because it was not unitary or did not preserve state normalization. This is most likely due to a difference in instruction; the question asked, with intentionally broad phrasing, “how might you construct a quantum gate...” The QCQI students may have focused on *how* to construct a gate and what such gates accomplish while Physics students may have taken the question to be asking them to construct and present an appropriate gate in matrix notation. Given that much of the Physics coursework involves matrix manipulations, they may have viewed this to be the most suitable course of action based upon the question prompt.

For Question 2b (describing a gate that transforms a given state to a basis state), some students, upon seeing that their projective gate  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ , when applied to the given state  $\sqrt{\frac{11}{13}}|0\rangle + \sqrt{\frac{2}{13}}|1\rangle$ , resulted in the state  $\sqrt{\frac{11}{13}}|0\rangle$ , attempted to achieve a normalized state by multiplying their provided gate by  $\sqrt{\frac{13}{11}}$ . While our grading rubric did not give such responses full credit, we see that they are applying the rules they learned to situations not explicitly discussed in the tutorial in the best ways that they know how. A few students gave exemplary responses for this question by specifying that the gate must be unitary and providing the additional constraints that render the problem soluble. Such students received full marks regardless of whether they were able to numerically complete the calculation to arrive at a suitable quantum gate. About one or two students from each class provided such answers for both Question 2b and 3d; while the remaining students did not articulate such concepts with the same precision, many QCQI students did make some mention of a “rotation,” referring to the fact that quantum gates are rotations of quantum states in a Hilbert space. This was likely to have been emphasized more in the QCQI course than the Physics course.

An additional point can be made about Question 7b, a two-part question that asks students what the probability amplitudes of a state  $a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$  must be if one desires to read an output of 01, and how this can be achieved if the initial state of the quantum computer did not have them. Most students successfully provided the probability amplitudes but did not answer the second part on the pre-test; a few who did answer said that this was impossible. The expected answer is that quantum gates can be applied to change the state of the system from an initial to a final state. On the post-test, there was substantial improvement in both completion and correctness on this question (see Tables 7.1-7.2), with most students citing quantum gates in some way. A few instead suggested, apparently assuming an infinite ensemble of quantum computers in superpositions of all four basis states, something to the effect of continuously sampling from the ensemble until one obtained the desired state as the output. Though these methods in some respects defeat the purpose of running a quantum computation, they were accepted as valid answers.

## **7.4.2 Difficulties assessed with a subset of students (excluding one class of Physics students)**

### **7.4.2.1 Outcomes of measurement**

Students did relatively well on Questions 9a-c on the pre-test, which asked them to provide the possible outcomes of various hypothetical measurements on given states (these questions were not posed to one of the Physics classes' pre-test and post-test as noted in Table 7.1). They improved to nearly full accuracy on the post-test (see Tables 7.1-7.2). These questions were similar to 2a and 3a, but with consecutive measurements. The most common mistakes involved not recognizing that the state would collapse after measurements.

#### 7.4.2.2 Finding the bra state corresponding to the given ket state

On the pre-test for Question 10a, which asked for the corresponding bra state to the given ket state, students did reasonably well, but they improved to near full accuracy on the post-test. Splitting the QCQI students between physics and non-physics majors reveals that the physics students were overall more comfortable with bra states on the pre-test, likely because many of them had already taken QM (see Appendix F.3). The most common mistake was not finding the complex conjugate of the expansion coefficients. Some students left this question blank.

#### 7.4.2.3 Difficulties with outer products, operators, and Dirac notation

For pre-test Question 10b, which gave students an operator  $a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$  and asked them to fill in the values of  $a, b, c, d$  for the identity operator, most students were unable to do so correctly. For pre-test Question 10c, many students were unable to provide a fully correct outer product. For both questions, many of the student responses gave the same value for all four matrix elements, which is likely to be a guess in the absence of better options. This could be connected to many students' tendency to say that all four outcomes are equiprobable in Question 1d. Others left the question blank. Given this apparent lack of fluency, it is unclear also whether the students would have had an easier time had they been asked to write their answer in matrix representation instead. In a typical physics course, students have been found to have difficulties distinguishing an inner product from an outer product, but this is almost universally corrected after engagement with a research-based learning tool, like a clicker question sequence or a tutorial [137]. On a related note, Question 10c asked students to find the outer product  $|q\rangle\langle q|$ ; some students stated that the identity operator was  $\hat{I} = |q\rangle\langle q|$ , which appears to be conflating it

with the spectral decomposition  $\hat{I} = \sum |q_i\rangle\langle q_i|$  where  $i$  enumerates each linearly independent basis state.

For Question 10c, which was similar to Question 10b but asked students to find the values of  $a, b, c, d$  for the outer product  $|q\rangle\langle q|$  for a given  $|q\rangle$ , there was some major improvement on the post-test, from 47% to 74%. Question 10b, on the other hand, saw little improvement (see Tables 7.1-7.2). It is possible that students did not have much practice with outer products after being exposed to them.

Question 11 asked students to write the Hadamard gate in Dirac notation. Students in the QCQI course who were physics majors may have had less difficulty if they had already taken QM, but it appeared to be quite difficult overall. Though some students answered the question correctly at least once, many left the question blank on the pre-test, or admitted to not knowing what the question was asking, and continued to struggle on the post-test. Answers on both the pre-test and post-test varied, and included ket states like  $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ , which may indicate that these students were not comfortable with the distinction between ket states and outer products. Some students provided the correct answer in the form  $\frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$ , but others responded with  $|0\rangle\langle +| + |1\rangle\langle -|$  or  $|+\rangle\langle 0| + |-\rangle\langle 1|$  which were not used in the QCQI course text and appeared to be a notable instance of student creativity. Overall, there was some improvement from the pre-test to the post-test, but the quality of answers was not uniform.

A summary of the difficulties discussed in this section can be found in Table 7.3.

**Table 7.3 Summary of student difficulties as well as students' improvement on these difficulties between the pre-test and post-test. Difficulties and associated questions that were assessed with only a subset of the students are marked with an asterisk (\*).**

Difficulties	Pre-/post-test #   comments
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Enumeration of all possible states or linearly independent states for a system of two bits or qubits, respectively	1a, 1b   High pre-test performance
Specifying outcomes of measurements made on qubits, including consecutive measurements made in the same or a different basis	2a, 3a, 3b, 3c, 9a*, 9b*, 9c*   High pre-test performance
Correct application of the Hadamard gate (given to students) to specified single-qubit states	3c   High pre-test performance
Inability to provide a physical example of a qubit	1c   Major improvement
Probabilities of measuring each possible outcome	1d   Some improvement; high pre-test performance
Difficulties with reasoning about various possible states of a quantum computer	7a, 7b, 7c   Some improvement
Difficulties with $N$ vs. $2^N$ entities	4a, 4b, 4c, 5, 6   Major improvement
Conflation of superposition and entanglement	4d, 8a, 8b, 8c, 8d   Major improvement in all cases except in cases of high pre-test performance (QCQI 8a, 8b, 8c; Physics 8c)
Difficulties with what single-qubit quantum gates accomplish	2b, 3d, 7b   Some improvement
*Finding the bra state corresponding to the given ket state	10a*   Some improvement
*Difficulties with outer products and operators	10b*, 10c*   Some improvement (greater for 10c)
*Difficulties with expressing quantum gates in Dirac notation	11*   Major improvement

### 7.4.3 End-of-semester retention

In addition to the pre- and post-test data, we also collected retention data for some questions for one of the physics classes ( $N = 22$ ). Details can be found in Appendix Table 2 in Appendix F.3. Students were evaluated on the concepts covered in Questions 4a-d, 5, and 10b on the final exam, which was given several months after the material was covered and the tutorial was assigned. All of these questions dealt with concepts regarding quantum computing except for 10b, which asked them to enumerate the matrix elements of the identity operator. Overall, students did

very well on these questions. Question 5 asked about the number of states (for classical computers) or linearly independent states (for quantum computers) available during a calculation; the somewhat lower performance here could be the result of the question presenting students with only one statement to evaluate rather than two statements to compare, which could possibly reduce the salience of the reasoning primitive that a classical computer has  $N$  available states compared to a quantum computer's  $2^N$  states.

Only a marginal improvement on 10b indicates that the identity operator given as an outer product and identification of suitable coefficients continues to be challenging for some students, with the most popular incorrect answer of all matrix elements being equal to one another still noticeably present. This seems to be a strong alternative conception when this question is framed in terms of outer products. Prior investigation shows that a comparable percentage of Physics students struggled when prompted to write the identity operator as an outer product of a complete set of orthonormal states, even on the post-test. Though Question 10b instead provided the generic outer product for which students were to identify the correct coefficients  $a-d$ , this indicates that students can routinely have difficulties with the identity operator expressed in Dirac notation [62]. On the final exam, two students multiplied the operator by an overall factor as though normalizing a ket state. This is curious, as it was observed only twice throughout the pre-test and post-test across all classes ( $N = 63$ ), so it appeared with higher frequency on the final ( $N = 22$ ). Additionally, one student wrote the identity operator in matrix form for a four-dimensional (instead of two-dimensional) Hilbert space. Some of the other students who noted that all four coefficients should be 1 may have also been thinking about an identity matrix in four-dimensional Hilbert space, possibly assuming that they were giving coefficients for only on-diagonal elements as opposed to coefficients for a  $2 \times 2$  matrix with both on-diagonal and off-diagonal elements.

## 7.5 Conclusions and discussion

### 7.5.1 Largest differences between QCQI and Physics students

It is somewhat surprising that the QCQI students performed better on many pre-test questions than Physics students, though in retrospect there is sense to this. Our initial expectations were that Physics students might have a better grasp of the underlying QM formalism and thus may be able find many salient connections to the material. However, we conclude that the QCQI course focused on those aspects in much more depth with no distraction from other type of course material than a typical quantum physics course for physics juniors/seniors would, even though both physics instructors teaching the latter course noted that they had covered the materials via lecture-based instruction. In particular, the QCQI students did better overall possibly because they had focused on this types of material (e.g., on qubits, multi-qubit systems, gates) for the entire semester and the topic of basics of quantum computing was not an addition to the typical material covered as in the junior/senior level Physics course.

Questions 2b, 3d, 4a, 4d, and 6 are ones for which the post-test performance differed by more than 15% between the Physics and QCQI students, with QCQI students scoring higher for all cases (see Tables 7.1-7.2). Questions 2b and 3d both dealt with conceptual understanding of quantum gates, giving students a starting state and asking them to comment on possible gates that could be applied to reach a desired state. Physics students were more likely to explicitly come up with a gate to apply to the state while QCQI students provided more conceptual statements. Since calculating a valid gate was a complicated process, and most students' numerical responses in both

courses did not succeed at doing so, this may have put the Physics students at a disadvantage. Questions 4a and 6 asked students whether particular enumerations in classical and quantum computers were  $N$  or  $2^N$ , which was another difficult concept for both populations on the pre-test, but one on which the QCQI students likely spent much more extensive class time. Question 4d asked students to explain the difference, if any, between superposition states and entangled states, and once again, this was a key concept that the QCQI course focused on.

In the data shown in Tables 7.1-7.2, the QCQI students' performance decreased by more than 10% when considering all students (unmatched data) for Questions 2b and 11. Given that the unmatched data comprise an additional 1 physics major and 9 non-physics majors, it follows that the vast majority of this decrease is contributed by non-physics majors. Question 11 dealt with Dirac notation, so non-physics majors in the course may have had more difficulty, or less prior knowledge, concerning quantum gates (Question 2b) and Dirac notation, particularly as it concerns outer products, which was the most difficult concept for students. This is investigated further in the following section, "*Comparison between physics and non-physics majors in the QCQI course.*"

With all this said, both QCQI and Physics students improved on these questions from the pre-test to the post-test. However, it was a trend in both courses that if the starting score was too low, about 50% or under, it ultimately did not rise to encompass the vast majority of the class as was observed in some other cases—such questions with low pre-test scores saw an improvement to roughly half correctness on the post-test. This is in line with previous observations that these tools work best if there is enough knowledge in the class already, which can be a combination of students' preparation, what they were able to learn from lecture, or other factors [82]. When students were answering questions on the pre-test with roughly half correctness, they were almost always able to improve to near full correctness after they engaged with the tutorial.

### 7.5.2 Comparison between physics and non-physics majors in the QCQI course

A bit of additional insight may be derived by comparing, within the QCQI course, the post-test performance of the physics majors to that of the non-physics majors. These results can be found in Appendix F.3.

When comparing the matched data, discrepancies of 15% or greater are observed in Questions 1c, 2b, 10c, and 11. When considering all of the students through the unmatched data, discrepancies of 15% or greater are observed in Questions 2b, 3a, 4c, 8d, 10c, and 11. Questions 2b, 10c and 11 are common to both the large and small datasets, with physics majors doing better, so in general it appears that gates, outer products, and Dirac notation are concepts that physics majors have an easier time with than non-physics majors. It is worth noting that roughly half of the physics majors in this course had completed the first semester of their upper-level QM course, which emphasizes these topics in the curriculum. Physics majors also did slightly better on Questions 4c (a question targeting the incorrect notion that the limit on building large quantum computers is the inability to contain the data that quantum computers handle) and 8d (a question on entanglement). Though these concepts are not as widely taught in a physics-focused QM course, physics majors' performance could still be related to familiarity with and time spent on related ideas in QM.

Some differences in post-test performance can also be seen between the physics and non-physics majors with Questions 1c and 3a in either the matched and unmatched data. As is the case with Questions 4c and 8d, however, the performance of both groups was high, and the difference amounts to a total of roughly two students performing better on the part of one group or the other.

Some differences were also observed between matched and unmatched non-physics major data, with drops of more than 10% for Questions 1c, 4d, 8d and 11 once all the non-physics majors

are considered. This suggests that the non-physics majors as a whole are less comfortable than physics majors with physical examples of qubits, the difference between superposition and entanglement, and Dirac notation.

### 7.5.3 Overall trends and impact

There were several concepts that were more challenging than others on the pre-test after traditional lecture-based instruction. These concepts included contending with the numbers  $N$  and  $2^N$  in the contexts of classical and quantum computers, e.g., in terms of how many operations must be performed to initialize each type of computer (seen in Questions 4a-c, 5, and 6) due to the reasoning primitive discussed. In particular, we identified a reasoning primitive [140] many students use for why a quantum computer may have a processing and storage advantage: these advantages come from the fact that certain entities for an  $N$ -bit classical computer get replaced with  $2^N$  for a quantum computer. This reasoning primitive manifested in many key concepts, leading many students to incorrectly conclude that a  $N$ -bit classical computer only has  $N$  available states for computation while an  $N$ -qubit quantum computer requires  $2^N$  entities to be initialized at the beginning of the computation, or that an  $N$ -qubit quantum computer works with  $2^N$  bits of information during the quantum computation or when yielding its output after measurement. Another challenging theme was the nature and role of quantum gates, including the importance of unitarity and how they are used to manipulate qubits into a desired state (seen in Questions 2b, 3e, and 7b). Students also experienced challenges with the difference between superposition and entanglement (seen in Questions 4b and 8a-d). Students also initially had some difficulty providing physical examples of qubits (seen in Question 1c), though most of these were a matter of not

specifying the two basis states of the qubit (while most did specify two states for their examples of a classical bit).

Students performed well on the vast majority of the post-test questions. The few questions for which students in each course scored under 70% were nearly all discussed in the preceding section, “*Largest differences between QCQI and Physics students,*” which in addition to being challenging questions, were also ones with the largest disparities between courses. The one additional question for which post-test performance is under 70% is Question 10b, which asked students to provide the matrix elements of the identity operator. Students in the QCQI course, especially non-physics majors, seemed to have some more difficulty with operators in general.

Overall, the post-test scores were not very different and were in fact quite comparable in the two courses after engaging with the tutorial. Thus, the tutorial appears to be beneficial for students in both courses. This suggests that, even with different instructors and different course contexts and broader goals, a research-based tutorial can be useful in helping students learn the basics of quantum computing. With the need for effective instruction to help students understand and communicate challenging foundational concepts in QISE, such a tutorial can play a key role in fulfilling this promise.

## **7.6 Acknowledgments**

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## 8.0 Future directions

In the previous chapters, I have outlined the development, validation, implementation, and evaluation of active learning instructional tools. The CQSs and tutorials have undergone many iterations with multiple years of administration and data. Altogether, the implementations have successfully raised student performance on the post-test and later retention assessments. However, the common difficulties of a few concepts, discussed in the previous chapters, remain somewhat unresolved. More effort and care should be devoted to the teaching of these concepts within these learning tools. It is unclear what the scale of change is necessary, or whether such difficulties can be resolved with time or require more incubation time for students to successfully address, but this can be investigated with further research. The materials can undergo further refinement, including consolidation or clear guidelines or procedures to aid instructors' efforts in customizing their suite of materials. Implementations at other institutions can similarly highlight other trends or needs that can be addressed and improved; institutions whose demographic makeup differs from that of a large research university in the United States can provide indications for how the material should be adapted to better serve various populations of students.

Invoking and common student difficulties and popular alternative conceptions is an effective and powerful way to engage students in the learning process and strike at the heart of potential confusion. The promising results of these materials speaks to the strength of this approach. The methods and techniques used for content in QM and QISE courses can be applied to that of other advanced courses, not only in physics but in various STEM fields with which physics shares many structural similarities. Obvious parallels can be drawn to the teaching of classical electrodynamics and thermodynamics/statistical mechanics, but advanced topics in

chemistry and biology (including notoriously challenging courses like physical chemistry, organic chemistry, and biochemistry, to name a few) can also stand to benefit greatly from the structure and approach of guided inquiry and peer-collaborative instruction.

In these investigations, I conducted a preliminary probe of content retention at the end of the semester. The results are highly encouraging, and more thorough retention studies can be carried out to further support the long-term benefits of these materials.







Use the following matrix representations

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Choose all of the following statements that are true:

- I.  $|+x\rangle\langle -x| = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$
- II.  $|+x\rangle\langle -x| = \frac{1}{2}(1 - 1 + 1 - 1) = 0$
- III.  $|+z\rangle\langle +z| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

I only      B. II only      C. III only      D. I and III only      E. None of the above

### CQS 2.6

Use the following matrix representations

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$|+y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad |-y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Choose all of the following statements that are true:

- I.  $\langle +x|+x\rangle = \frac{1}{\sqrt{2}} (1 \ 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$
- II.  $\langle +x|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} (1 \ 1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
- III.  $|+z\rangle\langle -x| = (1 \ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}$
- IV.  $|+z\rangle\langle -x| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} (1 \ -1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

- A. I only
- B. I and III only
- C. I and IV only
- D. II and III only
- E. II and IV only

### CQS 3.1

A generic state is written in the  $\{|+z\rangle, |-z\rangle\}$  basis as  $|\chi\rangle = a|+z\rangle + b|-z\rangle$ , but it can be written in another basis. In the  $\{|+y\rangle, |-y\rangle\}$  basis,  $|\chi\rangle = a'|+y\rangle + b'|-y\rangle$ .

Choose all of the following statements that are true:

- I.  $a = \langle +z|\chi\rangle$  and  $b = \langle -z|\chi\rangle$
- II.  $a'$  and  $b'$  are the projections of  $|\chi\rangle$  along  $|+y\rangle$  and  $|-y\rangle$ , respectively.
- III.  $a' = \langle +y|\chi\rangle$  and  $b' = \langle -y|\chi\rangle$

A. I only      B. II only      C. I and III only      D. II and III only      (All of the above)

### CQS 3.2

A generic state is written in the  $\{|+z\rangle, |-z\rangle\}$  basis as  $|\chi\rangle = a|+z\rangle + b|-z\rangle$ . Given that  $|+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + i|-z\rangle)$  and  $|-y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - i|-z\rangle)$ , choose all of the following statements that are true:

- I.  $|+z\rangle = \frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle)$  and  $|-z\rangle = -\frac{i}{\sqrt{2}}(|+y\rangle - |-y\rangle)$   
 II.  $|\chi\rangle = \frac{a}{\sqrt{2}}|+y\rangle + \frac{ib}{\sqrt{2}}|-y\rangle$   
 III.  $|\chi\rangle = a\left[\frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle)\right] + b\left[-\frac{i}{\sqrt{2}}(|+y\rangle - |-y\rangle)\right]$

A. I only      B. II only      C. III only      D. I and II only      E. I and III only

Notes to Instructor for CQS 3.2:

- Give students ~2 minutes for CQS 3.2, to calculate what the  $|\pm z\rangle$  states are in terms of the  $|\pm y\rangle$  states.
- Show that the expansion coefficients in the y-basis ( $a'|+y\rangle + b'|-y\rangle$ ) simplify to  $a' = \frac{a-bi}{\sqrt{2}}$  and  $b' = \frac{a+bi}{\sqrt{2}}$ , where  $a$  and  $b$  are the expansion coefficients in the z-basis,  $a|+z\rangle + b|-z\rangle$ .
- Discuss the special case when  $a = \frac{1}{\sqrt{2}}$  and  $b = \pm \frac{i}{\sqrt{2}}$ . These yield the eigenstates of the y-basis, so that of  $a'$  and  $b'$ , one would be 0 and the other would be 1.

**CQS 3.3**

One way to accomplish the process of writing a state in a basis is by acting upon it with the operator  $\hat{I}$  written in that basis. Choose all of the following that are valid statements:

- I.  $\hat{I} = |+x\rangle\langle+x|$   
 II.  $\hat{I} = |+x\rangle\langle+x| + |-x\rangle\langle-x|$   
 III.  $\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  in any basis chosen  
 IV.  $\hat{I}|\chi\rangle = |\chi\rangle$

A. I and III only      B. III and IV only      C. I, III, and IV only  
 D. II, III, and IV only      E. All of the above

Notes to Instructor for CQS 3.3

- Discuss that the identity operator is the sum of the outer products of a complete set of orthonormal basis states.
- Remind students that the identity operator  $\hat{I}$  acting on any state returns that same state.  $\hat{I}$  can be placed anywhere based upon convenience, and is analogous to multiplying a scalar by 1.
- Show (or have students confirm) that the sum of the outer products for a complete set of orthonormal basis vectors returns the identity operator in matrix form in any basis.

**CQS 3.4**

Given that the identity operator  $\hat{I}$  multiplied by any state returns that state, choose all of the following that are an equivalent way of writing the state  $|\chi\rangle$ :

- I.  $|\chi\rangle = \hat{I}|\chi\rangle = (|+x\rangle\langle+x| + |-x\rangle\langle-x|) |\chi\rangle$   
 II.  $|\chi\rangle = \hat{I}|\chi\rangle = (|+y\rangle\langle+y| + |-y\rangle\langle-y|) |\chi\rangle$   
 III.  $|\chi\rangle = \hat{I}|\chi\rangle = (|+x\rangle\langle+x| + |-y\rangle\langle-y|) |\chi\rangle$

A. I only      B. II only      C. III only      D. I and II only      E. All of the above

### CQS 3.5

If we want to express our state in the x basis, we can write the following:

$$|\chi\rangle = \hat{I}|\chi\rangle = (|+x\rangle\langle+x| + |-x\rangle\langle-x|) |\chi\rangle$$

Choose all of the following that correctly represent the state  $|\chi\rangle = a|+z\rangle + b|-z\rangle$  in the  $\{|+x\rangle, |-x\rangle\}$  basis:

- I.  $|\chi\rangle = (|+x\rangle\langle+x| + |-x\rangle\langle-x|) (a|+z\rangle + b|-z\rangle)$   
 II.  $|\chi\rangle = |+x\rangle\langle+x|a|+z\rangle + |-x\rangle\langle-x|b|-z\rangle$   
 III.  $|\chi\rangle = a|+x\rangle + b|-x\rangle$

A. I only      B. II only      C. III only      D. All of the above      E. None of the above

## Appendix A.2 Pre-test and post-test questions

The post-test questions are reproduced below. Pre-test questions are isomorphic with small changes, such as different given states. Students were given the following information:

- Whenever a state is written in matrix form in a given basis as opposed to Dirac notation, the symbol  $\hat{=}$  (which stands for “is represented by”) is substituted by an equals sign, =, for convenience. For example,  $|+z\rangle \hat{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  will be written as  $|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ .
- If you are familiar with the states in the notation  $|\uparrow\rangle_z$  &  $|\downarrow\rangle_z$ ,  $|\uparrow\rangle_x$  &  $|\downarrow\rangle_x$ , and  $|\uparrow\rangle_y$  &  $|\downarrow\rangle_y$ , they are the same as the states  $|z\rangle$  &  $|-z\rangle$ ,  $|x\rangle$  &  $|-x\rangle$ , and  $|y\rangle$  &  $|-y\rangle$ , respectively, such that:

$$\begin{aligned} \hat{S}_z|+z\rangle &= \frac{\hbar}{2}|+z\rangle & \hat{S}_z|-z\rangle &= -\frac{\hbar}{2}|-z\rangle \\ \hat{S}_x|+x\rangle &= \frac{\hbar}{2}|+x\rangle & \hat{S}_x|-x\rangle &= -\frac{\hbar}{2}|-x\rangle \\ \hat{S}_y|+y\rangle &= \frac{\hbar}{2}|+y\rangle & \hat{S}_y|-y\rangle &= -\frac{\hbar}{2}|-y\rangle \end{aligned}$$

In the z-basis, the basis states  $|\pm z\rangle$ ,  $|\pm x\rangle$  and  $|\pm y\rangle$  can be obtained by diagonalizing  $\hat{S}_z$ ,  $\hat{S}_x$ , and  $\hat{S}_y$ , respectively, where:

$$\begin{aligned} \hat{S}_z &= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \hat{S}_x &= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \hat{S}_y &= \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{aligned}$$

- $\langle \phi | \chi \rangle = \langle \chi | \phi \rangle^*$  for any two generic states  $|\chi\rangle$  and  $|\phi\rangle$ , where  $*$  denotes the complex conjugate:  $(a \pm ib)^* = (a \mp ib)$
- The eigenstates of  $\hat{S}_x$  in terms of the eigenstates of  $\hat{S}_z$  are as follows:

$$|+x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$$

$$|-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle)$$

1. Consider the following statements about a 2-dimensional vector space in which the state of a quantum system lies. Are the following statements true or false?

\_\_\_ a) For every physical observable, there is a corresponding Hermitian operator that can act on states in the vector space.

\_\_\_ b) Once you have chosen a basis, you can represent any operator as a  $2 \times 2$  matrix.

\_\_\_ c) Once you have chosen a basis, you can represent any ket state as a  $2 \times 1$  column matrix.

2. Given that  $|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $|-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Write  $|+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + i|-z\rangle)$  and  $|-x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle)$  as matrices in this basis.

3. Given that in the  $z$  basis

$$|\chi\rangle = \frac{1}{\sqrt{34}} \begin{pmatrix} 3i \\ 5 \end{pmatrix} \qquad |\psi\rangle = \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ -i \end{pmatrix}$$

Represent  $|\chi\rangle$  and  $|\psi\rangle$  in the form  $|\chi\rangle = c|+z\rangle + d|-z\rangle$ ,  $|\psi\rangle = e|+z\rangle + f|-z\rangle$  (that is, find  $c$ ,  $d$ ,  $e$ , and  $f$ ).

4. Given that in the  $z$ -basis,  $|\chi\rangle = \frac{1}{\sqrt{34}} \begin{pmatrix} 3i \\ 5 \end{pmatrix}$  and  $|\psi\rangle = \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ -i \end{pmatrix}$ , find  $\langle \chi | \psi \rangle$ . Show or explain your work to get credit.

5. Given that in the  $z$ -basis,  $|\chi\rangle = \frac{1}{\sqrt{34}} \begin{pmatrix} 3i \\ 5 \end{pmatrix}$  and  $|\psi\rangle = \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ -i \end{pmatrix}$ , find  $|\psi\rangle\langle\chi|$ . Show or explain your work to get credit.

6. Write the state  $|\chi\rangle = \frac{3}{\sqrt{10}}|+x\rangle + \frac{1}{\sqrt{10}}|-x\rangle$  in the form  $|\chi\rangle = a'|+z\rangle + b'|-z\rangle$  (that is, find  $a'$  and  $b'$ ). Show or explain your work.

7. Write the state  $|\chi\rangle = \frac{3}{\sqrt{10}}|+z\rangle + \frac{1}{\sqrt{10}}|-z\rangle$  in the form  $|\chi\rangle = a'|+x\rangle + b'|-x\rangle$  (that is, find  $a'$  and  $b'$ ). Show or explain your work.

Note that we start here in the  $z$ -basis and wish to convert to  $x$ -basis, the opposite of the preceding question.

### Appendix A.3 Retention questions

The questions on the midterm that corresponded to selected questions on the pre- and post-test are reproduced below.

2. Given that  $|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $|-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Write  $|+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + i|-z\rangle)$  and  $|-y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - i|-z\rangle)$  as matrices in this basis.

6. Consider a state  $|\chi\rangle = \frac{3}{5}|+y\rangle + \frac{4}{5}|-y\rangle$ . If you measure  $S_z$ , what is the probability of obtaining a value  $\frac{\hbar}{2}$ ?

7. Write the state  $|\chi\rangle = \frac{3}{\sqrt{10}}|+z\rangle + \frac{1}{\sqrt{10}}|-z\rangle$  in the form  $|\chi\rangle = a'|+y\rangle + b'|-y\rangle$  (that is, find  $a'$  and  $b'$ ). Show or explain your work.

## Appendix B Materials for Chapter 3

### Appendix B.1 Clicker questions

The clicker questions in the sequence are reproduced below.

- All of the questions refer to a two-state system.
- The notation  $|\chi(0)\rangle$  indicates the state  $|\chi\rangle$  at time  $t = 0$ , and  $|\chi(t)\rangle$  indicates the state  $|\chi\rangle$  at time  $t$ .
- The eigenstates of the operator  $\hat{S}_z$  are written as  $\{|z\rangle, |-z\rangle\}$ , those of  $\hat{S}_x$  are written as  $\{|x\rangle, |-x\rangle\}$ , and those of  $\hat{S}_y$  are written as  $\{|y\rangle, |-y\rangle\}$ . The energy eigenvalues for the two-state system are denoted by  $E_+$  and  $E_-$ .
- The Hamiltonian, and all other observables of interest, are assumed to have no explicit time-dependence.
- When a two-state system is written as  $a|\chi_1\rangle + b|\chi_2\rangle$ , it is assumed that  $|a|^2 + |b|^2 = 1$ ,  $|a| \neq 0$  and  $|b| \neq 0$ , and  $a \neq b$ .

#### (CQS 1.1)

Learning objective: Students are able to identify the properties of stationary states.

Choose all of the following statements that are correct:

- The term “stationary state” refers to an eigenstate of any operator that corresponds to a physical observable.
  - The term “stationary state” refers to an eigenstate of the Hamiltonian only.
  - A system in a stationary state will stay in this state for all times  $t$  unless it is externally perturbed.
- A. I only                                      B. II only                                      C. III only  
D. I and III only                              E. II and III only

#### (CQS 1.2)

Learning objective: Students are able to identify the properties of stationary states.

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ . Choose all of the following initial states  $|\chi(0)\rangle$  that are stationary states:

- $|\chi(0)\rangle = |z\rangle$
  - $|\chi(0)\rangle = |x\rangle$
  - $|\chi(0)\rangle = a|z\rangle + b|-z\rangle$ , because  $|z\rangle$  and  $|-z\rangle$  are both stationary states.
- A. All of the above                              B. I only                                      C. I and II only  
D. I and III only                              E. None of the above

#### Class discussion for CQS 1.1-1.2

- The eigenstates of the Hamiltonian are known as stationary states and have trivial time evolution, in which the state at time  $t$  depends upon an overall time-dependent phase factor, e.g.,  $|\chi(t)\rangle = e^{-\frac{iEt}{\hbar}}|\chi(0)\rangle = e^{-\frac{iE_+t}{\hbar}}|z\rangle$  if  $|\chi(0)\rangle = |z\rangle$  is a stationary state.
- A superposition of stationary states is not a stationary state if there is no degeneracy.

**(CQS 2.1)**

*Learning objective: Students are able to make a distinction between eigenstates of the Hamiltonian (and operators that commute with the Hamiltonian) and eigenstates of any other operator.*

Note: Any observable (which, for generality, we'll call  $Q$ ) has a corresponding Hermitian operator  $\hat{Q}$ , which has a complete set of eigenstates  $|q_+\rangle$  and  $|q_-\rangle$  with eigenvalues  $q_+$  and  $q_-$  respectively. Assume that  $\hat{Q}$  does not commute with the Hamiltonian.

Choose all of the following that are correct about the time-development of a generic state  $|\chi(0)\rangle$ :

- I. The time evolution of the state is governed by the Hamiltonian operator for that system.
  - II. If  $|\chi(0)\rangle$  is an energy eigenstate (also known as a stationary state), it will remain an energy eigenstate after a time  $t$ .
  - III. If  $|\chi(0)\rangle$  is an eigenstate of  $\hat{Q}$ , it will remain an eigenstate of  $\hat{Q}$  after a time  $t$ .
- A. All of the above                      B. I only                      C. II only  
 D. I and II only                      E. II and III only

**(CQS 2.2)**

*Learning objective: Students are able to describe the effect of the time-evolution operator when applied to a generic non-stationary state.*

Choose all of the following that are correct about the time-development of a state  $|\chi(0)\rangle$ , which in this case is **not a stationary state**:

- I.  $|\chi(t)\rangle = e^{\frac{-iE_+t}{\hbar}} |\chi(0)\rangle$  or  $e^{\frac{-iE_-t}{\hbar}} |\chi(0)\rangle$ , where  $E_+$  and  $E_-$  are eigenvalues of  $\hat{H}$ .
  - II.  $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} |\chi(0)\rangle$ , where  $\hat{H}$  is the Hamiltonian of the system.
  - III. The probability of measuring  $q_+$  and  $q_-$  in the state  $|\chi(t)\rangle$  will be the same, regardless of the time  $t$  when the measurement is performed. [51]
- A. II only                      B. III only                      C. I and II only  
 D. II and III only                      E. None of the above

**(CQS 2.3)**

*Learning objective: Students are able to describe the effect of the time-evolution operator  $e^{\frac{-i\hat{H}t}{\hbar}}$  when applied to a concrete example of an initial state which is a non-stationary state not written as a superposition.*

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ . Choose all of the following that are correct at time  $t$  about the system if the initial state is  $|\chi(0)\rangle = |y\rangle$ :

- I.  $|\chi(t)\rangle = e^{\frac{-iE_+t}{\hbar}} |\chi(0)\rangle$
  - II.  $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} |\chi(0)\rangle$
  - III. If the observable  $S_y$  is measured, the probability of obtaining  $\frac{\hbar}{2}$  is 1, regardless of time  $t$  when the measurement is performed.
- A. All of the above                      B. I only                      C. II only  
 D. I and II only                      E. II and III only

**(CQS 2.4)**

*Learning objective: Students are able to describe that the effect of the time-evolution operator when applied to a stationary state is to evolve the state by an overall time-dependent phase factor (i.e., the state remains a stationary state).*

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ . Choose all of the following that are correct at time  $t$  about the system if the initial state is  $|\chi(0)\rangle = |z\rangle$ :

- I.  $|\chi(t)\rangle = e^{\frac{-iE_+t}{\hbar}} |\chi(0)\rangle$
- II.  $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} |\chi(0)\rangle$
- III. If the observable  $S_z$  is measured, the probability of obtaining  $\frac{\hbar}{2}$  is 1, regardless of time  $t$  when the measurement is performed.

- A. I only                                      B. I and II only                                      C. I and III only  
 D. II and III only                                      E. All of the above

**(CQS 2.5)**

*Learning objective: Students are able to describe the effect of the time-evolution operator when applied to a quantum state written as a superposition of the eigenstates of  $\hat{H}$ .*

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$  in the state  $|\chi(0)\rangle = a|z\rangle + b|-z\rangle$ . Choose all of the following expressions that are correct for the state  $|\chi(t)\rangle$  at time  $t$ :

- I.  $|\chi(t)\rangle = e^{\frac{-iE_+t}{\hbar}} (a|z\rangle + b|-z\rangle)$
- II.  $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} (a|z\rangle + b|-z\rangle)$
- III.  $|\chi(t)\rangle = ae^{\frac{-iE_+t}{\hbar}} |z\rangle + be^{\frac{-iE_-t}{\hbar}} |-z\rangle$

- A. I only                                      B. II only                                      C. III only  
 D. I and II only                                      E. II and III only

**Class discussion for CQS 2.1-2.5**

- The previous 5 questions may be asked consecutively, with a discussion at the end. They involve several different situations, from general cases to concrete examples.
- $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} |\chi(0)\rangle$  is always true, but  $|\chi(t)\rangle = e^{\frac{-iE_+t}{\hbar}} |\chi(0)\rangle$  (an overall phase factor) is true only if  $|\chi(0)\rangle$  is the eigenstate of the Hamiltonian corresponding to the energy  $|E_+\rangle$  or  $|E_-\rangle$ .

**(CQS 3.1)**

*Learning objective: Students are able to describe how the completeness relation (spectral decomposition of the identity operator) can be used to write a generic state in a basis consisting of energy eigenstates. Other learning objective are similar to those in CQS 2.5.*

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ . Choose all the correct statements about a system in the state  $|\chi(0)\rangle$ :

- I.  $|\chi(0)\rangle = (|z\rangle\langle z| + |-z\rangle\langle -z|)|\chi(0)\rangle$ , where  $|z\rangle\langle z| + |-z\rangle\langle -z| = \hat{I}$
- II.  $|\chi(0)\rangle = C_1|z\rangle + C_2|-z\rangle$ , where  $C_1 = \langle z|\chi(0)\rangle$  and  $C_2 = \langle -z|\chi(0)\rangle$
- III.  $|\chi(t)\rangle = e^{\frac{-iE_+t}{\hbar}} C_1 |z\rangle + e^{\frac{-iE_-t}{\hbar}} C_2 |-z\rangle$

- A. I only                                      B. II only                                      C. I and II only  
 D. II and III only                                      E. All of the above

**(CQS 3.2)**

*Learning objective: Students are able to identify that the state first must be written in the energy eigenbasis before the time-evolution operator can be used to write the state after time  $t$  in the energy eigenbasis without the operator  $\hat{H}$ .*

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ . Choose all of the following that are correct about a system in the state  $|\chi(0)\rangle = a|x\rangle + b|-x\rangle$ .

- I.  $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}}|\chi(0)\rangle$   
 II.  $|\chi(t)\rangle = ae^{\frac{-iE_+t}{\hbar}}|x\rangle + be^{\frac{-iE_-t}{\hbar}}|-x\rangle$   
 III. To find  $|\chi(t)\rangle$ , we can write  $|\chi(0)\rangle$  as a linear superposition of energy eigenstates, and then attach a time-dependent phase factor with the appropriate energy to each term.
- A. I only                                      B. III only                                      C. I and II only  
 D. I and III only                              E. All of the above

**(CQS 3.3)**

*Learning objective: Students are able to identify that the state must be in the energy eigenbasis before the time-evolution operator can be applied in order to obtain an explicit expression for the time-dependent state. When the given initial state is not written in terms of the energy eigenbasis, they are able to write the time-evolved state in this basis.*

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ . Choose the correct expression for the time-evolved state  $|\chi(t)\rangle$  given an initial state  $|\chi(0)\rangle = a|x\rangle + b|-x\rangle$ .

- A.  $|\chi(t)\rangle = ae^{\frac{-iE_+t}{\hbar}}|x\rangle + be^{\frac{-iE_-t}{\hbar}}|-x\rangle$   
 B.  $|\chi(t)\rangle = \frac{a}{\sqrt{2}}e^{\frac{-iE_+t}{\hbar}}|z\rangle + \frac{b}{\sqrt{2}}e^{\frac{-iE_-t}{\hbar}}|-z\rangle$   
 C.  $|\chi(t)\rangle = \frac{a+b}{\sqrt{2}}e^{\frac{-iE_+t}{\hbar}}|x\rangle + \frac{a-b}{\sqrt{2}}e^{\frac{-iE_-t}{\hbar}}|-x\rangle$   
 D.  $|\chi(t)\rangle = \frac{a+b}{\sqrt{2}}e^{\frac{-iE_+t}{\hbar}}|z\rangle + \frac{a-b}{\sqrt{2}}e^{\frac{-iE_-t}{\hbar}}|-z\rangle$   
 E. None of the above

**(CQS 3.4)**

*Learning objective: Students are able to identify that the state must be in the energy eigenbasis before the time-evolution operator can be applied in order to obtain an explicit expression for the time-dependent state.. Here,  $\hat{H} \propto \hat{S}_x$ , so a transformation from the z-basis to the x-basis is required.*

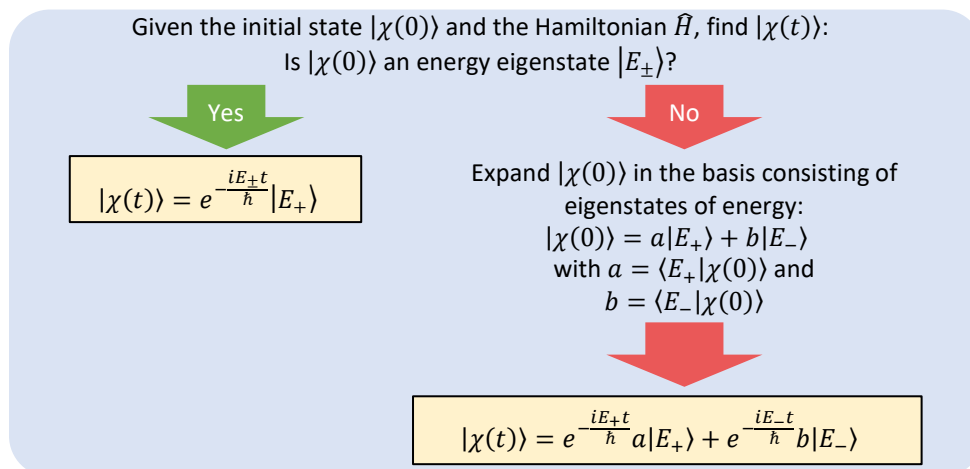
Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_x$ . Choose all of the following that are correct about the time-development of the state  $|\chi(0)\rangle = \frac{1}{\sqrt{2}}|z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$ :

- I.  $|\chi(t)\rangle = \frac{1}{\sqrt{2}}e^{\frac{-iE_+t}{\hbar}}|z\rangle + \frac{1}{\sqrt{2}}e^{\frac{-iE_-t}{\hbar}}|-z\rangle$   
 II.  $|\chi(t)\rangle = \frac{1}{\sqrt{2}}e^{\frac{-iE_+t}{\hbar}}|x\rangle + \frac{1}{\sqrt{2}}e^{\frac{-iE_-t}{\hbar}}|-x\rangle$   
 III.  $|\chi(t)\rangle = e^{\frac{-iE_+t}{\hbar}}|x\rangle$
- A. I only                                      B. II only                                      C. III only  
 D. I and III only                              E. None of the above

**Class discussion for CQS 3.1-3.4**

- The instructor can choose to give the following relations to the students for CQS 3.3-3.4.
 

$\langle z x\rangle = \frac{1}{\sqrt{2}}$	$\langle -z x\rangle = \frac{1}{\sqrt{2}}$
$\langle z -x\rangle = \frac{1}{\sqrt{2}}$	$\langle -z -x\rangle = -\frac{1}{\sqrt{2}}$



**Appendix Figure 1** A flowchart detailing the process of evolving a given initial state in time, provided to students as part of class discussion.

**(CQS 3.5)**

*Learning objective: Students are able to identify that the expectation values of energy and all observables whose corresponding operators commute with the Hamiltonian are independent of time, regardless of the initial state.*

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ . If the system is in the state  $|\chi(0)\rangle = a|z\rangle + b|-z\rangle$ , choose all of the following observables whose expectation values are time-independent:

- I. Energy
- II.  $S_z$
- III.  $S_x$

- A. None of the above
- D. II and III only

- B. I only
- E. All of the above

- C. I and II only

**(CQS 3.6)**

*Learning objective: Students are able to identify that the expectation values of all observables for a stationary state are independent of time.*

Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ . If the system is in the state  $|\chi(0)\rangle = |z\rangle$ , choose all of the following observables whose expectation values are time-independent:

- I. Energy
- II.  $S_z$
- III.  $S_x$

- A. None of the above
- D. II and III only

- B. I only
- E. All of the above

- C. I and II only

*Class discussion for CQS 4.1-4.2*

- The expectation value of energy is always time-independent regardless of the state (so long as  $\hat{H}$  does not explicitly depend on time).
- $S_z$  is special in this instance, because it is an observable whose corresponding operator commutes with the Hamiltonian. Its expectation value is also time-independent, regardless of the state.
- Expectation values of other observables such as  $\langle S_x \rangle$  and  $\langle S_y \rangle$  are time-dependent in CQS 4.1. But if the Hamiltonian were proportional to  $\hat{S}_x$ ,  $\langle S_x \rangle$  would be time-independent regardless of the state, while  $\langle S_z \rangle$  would be time-dependent.

## Appendix B.2 Pre-test and post-test questions

The pre- and post-test questions are reproduced below. The same information provided at the beginning of the clicker questions applies to the pre-test and post-test questions. Students were also given the following information:

$$\begin{aligned}\langle x|z\rangle &= \frac{1}{\sqrt{2}} & \langle -x|z\rangle &= \frac{1}{\sqrt{2}} \\ \langle x|-z\rangle &= \frac{1}{\sqrt{2}} & \langle -x|-z\rangle &= -\frac{1}{\sqrt{2}}\end{aligned}$$

Note: Any observable (which, for generality, we'll call  $Q$ ) has a corresponding Hermitian operator  $\hat{Q}$ , which has a complete set of eigenstates  $|q_+\rangle$  and  $|q_-\rangle$  with eigenvalues  $q_+$  and  $q_-$  respectively. Assume that  $\hat{Q}$  does not commute with the Hamiltonian.

**(Q1)** Choose all of the following that are correct about the time-development of a state  $|\chi(0)\rangle$ , which in this case is **not a stationary state**:

- I.  $|\chi(t)\rangle = e^{\frac{-iE_+t}{\hbar}} |\chi(0)\rangle$  or  $e^{\frac{-iE_-t}{\hbar}} |\chi(0)\rangle$ , where  $E_+$  and  $E_-$  are eigenvalues of  $\hat{H}$ .
- II.  $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} |\chi(0)\rangle$ , where  $\hat{H}$  is the Hamiltonian of the system.
- III. The probability of measuring  $q_+$  and  $q_-$  in the state  $|\chi(t)\rangle$  will be the same, regardless of the time  $t$  when the measurement is performed. [51]

**(Q2)** Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ , where  $C$  is an appropriate constant. Choose all of the following that are stationary states for this system:

- I.  $\frac{1}{\sqrt{2}}|z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$
- II.  $a|z\rangle + b|-z\rangle$
- III.  $a|x\rangle + b|-x\rangle$ , where  $a \neq b$

- (Q3) Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_x$  (note:  $\hat{H} \neq C\hat{S}_z$ ), where  $C$  is an appropriate constant. For a system in the initial state  $|\chi(t=0)\rangle = \frac{1}{\sqrt{5}}|x\rangle + \frac{2}{\sqrt{5}}|-x\rangle$ , what is  $|\chi(t)\rangle$ ? Show your work.
- (Q4) Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_x$ , where  $C$  is an appropriate constant. For a system in the initial state  $|\chi(0)\rangle = \frac{1}{\sqrt{5}}|z\rangle + \frac{2}{\sqrt{5}}|-z\rangle$ , what is  $|\chi(t)\rangle$ ? Show your work.
- (Q5) Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ , where  $C$  is an appropriate constant. For a system in the initial state  $|\chi(0)\rangle = a|z\rangle + b|-z\rangle$ , choose all of the following observables whose expectation values are time-independent:
- I. Energy
  - II.  $S_y$
  - III.  $S_z$
- (Q6) Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_z$ , where  $C$  is an appropriate constant. For a system in the state  $|\chi(0)\rangle = |z\rangle$ , choose all of the following observables whose expectation values are time-independent:
- I. Energy
  - II.  $S_y$
  - III.  $S_z$

## Appendix C Materials for Chapter 4

### Appendix C.1 Pre-test and post-test questions

The pretest and post-test questions are provided here.

Students were given the following information:

- The spin operators  $\hat{S}_z$ ,  $\hat{S}_x$ , and  $\hat{S}_y$  correspond to the observables  $S_z$ ,  $S_x$ , and  $S_y$ , respectively, which in turn correspond to the  $z$ ,  $x$ , and  $y$ -components of a particle's spin.
  - $\hat{S}_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$                        $\hat{S}_z|-z\rangle = -\frac{\hbar}{2}|-z\rangle$
  - $\hat{S}_x|+x\rangle = \frac{\hbar}{2}|+x\rangle$                        $\hat{S}_x|-x\rangle = -\frac{\hbar}{2}|-x\rangle$
  - $\hat{S}_y|+y\rangle = \frac{\hbar}{2}|+y\rangle$                        $\hat{S}_y|-y\rangle = -\frac{\hbar}{2}|-y\rangle$
- $|+z\rangle = \frac{1}{\sqrt{2}}|+x\rangle + \frac{1}{\sqrt{2}}|-x\rangle$                        $|-z\rangle = \frac{1}{\sqrt{2}}|+x\rangle - \frac{1}{\sqrt{2}}|-x\rangle$
- $|+x\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$                        $|-x\rangle = \frac{1}{\sqrt{2}}|+z\rangle - \frac{1}{\sqrt{2}}|-z\rangle$
- $a|+z\rangle + b|-z\rangle = \frac{a+b}{\sqrt{2}}|+x\rangle + \frac{a-b}{\sqrt{2}}|-x\rangle$
- $a|+x\rangle + b|-x\rangle = \frac{a+b}{\sqrt{2}}|+z\rangle + \frac{a-b}{\sqrt{2}}|-z\rangle$
- In all instances,  $\hat{H} = C\hat{S}_z$ , where  $C$  is a suitable constant. The energy eigenvalues are  $E_+$  and  $E_-$ .

1. Consider a system in the state  $|\chi\rangle = \frac{5}{13}|+z\rangle + \frac{12}{13}|-z\rangle$  with Hamiltonian  $\hat{H} = C\hat{S}_z$ . For simplicity, the energies corresponding to the  $|+z\rangle$  and  $|-z\rangle$  states are given as  $E_+$  and  $E_-$ , respectively. Is the following statement true or false?

$$\hat{H}|\chi\rangle = E_+|+z\rangle \text{ or } E_-|-z\rangle$$

2. Consider the normalized state  $|\chi\rangle = a|+y\rangle + b|-y\rangle$ . Are each of the following statements true or false?

\_\_\_\_\_ a) When an operator  $\hat{S}_y$  acts on state  $|\chi\rangle$ , it is equivalent to the measurement of the observable  $S_y$ , and the measurement process is given by

$$\hat{S}_y|\chi\rangle = \frac{\hbar}{2}|+y\rangle \text{ or } -\frac{\hbar}{2}|-y\rangle$$

with  $|\chi\rangle$  on the left-hand side representing the state before the measurement, and  $|+y\rangle$  or  $|-y\rangle$  on the right-hand side representing the state after the measurement.

\_\_\_\_\_ b)  $|a|^2 + |b|^2 = 1$

\_\_\_\_\_ c) If  $S_z$  (NOT  $S_y$ ) is measured, the probability of obtaining  $\frac{\hbar}{2}$  is  $|a|^2$ .

3. Consider a system in the state  $|\chi\rangle = \left(\frac{2}{5} - \frac{2}{5}i\right)|+x\rangle + \frac{\sqrt{17}}{5}| -x\rangle$ . If  $S_x$  is measured, what are the possible outcomes, and what are their probabilities?
4. Consider a state  $|\chi\rangle = \frac{4}{5}|+x\rangle + \frac{3}{5}| -x\rangle$  with Hamiltonian  $\hat{H} = \mathcal{C}\hat{S}_z$ . (You may wish to refer to the information provided at the beginning of the pretest/post-test for some useful relations.)
  - a) If you measure  $S_x$  and obtain a value  $\frac{\hbar}{2}$ , what is the normalized state of the system right after the measurement?
  - b) Immediately after you measure  $S_x$  and obtain  $\frac{\hbar}{2}$ , you measure  $S_x$  again. What is the probability of obtaining  $-\frac{\hbar}{2}$ ?
  - c) Immediately after you measure  $S_x$  and obtain  $\frac{\hbar}{2}$ , you measure  $S_z$ . What is the probability of obtaining  $-\frac{\hbar}{2}$ ?
  - d) Immediately after you measure  $S_x$  and obtain  $\frac{\hbar}{2}$ , you measure  $S_z$  and then  $S_x$  again, both in immediate succession. What is the probability of obtaining  $-\frac{\hbar}{2}$  for this last measurement of  $S_x$ ?
5. [Question asked in in-person class 2.]  
 Consider a system in the state  $|\chi\rangle = \left(\frac{2}{5} - \frac{2}{5}i\right)|+x\rangle + \frac{\sqrt{17}}{5}| -x\rangle$ . If many measurements of  $S_x$  are made on identical systems prepared in this state, what is the expectation value of those measurements? Show or explain your work.

### Appendix C.2 Clicker questions

The CQS questions are provided here. Correct answers are in bold type. As an additional note, the content that follows in these appendices is intended to help students understand the quantum measurement formalism, rather than technical details of quantum measurements in practice.

#### Notes to the instructor

- Students should be familiar with changing basis in two-state spin systems.
- CQS 1.1-1.3 revolve around the idea that measuring an observable in a given quantum state (a physical process involving an apparatus) is not the same as the corresponding operator acting on the state; specific cases are examined involving a spin operator, a Hamiltonian, and a generic operator  $\hat{Q}$ .
- CQS 2.1 helps students recognize that the expansion coefficients in a particular basis may be complex. CQS 2.2 emphasizes to students that the initial state must be written in the measurement basis associated with the observable to be able to determine the outcomes and the probabilities of measuring those outcomes. CQS 2.3 illustrates that, to reflect measurement collapse, the state after measurement must be normalized.
- CQS 3.1-3.2 focus on the collapse of a particular state into an eigenstate of the operator corresponding to the measured component of spin. They can help to reinforce that such eigenstates will result in 50/50 probabilities when either of the other two components of spin is subsequently and immediately measured.

- CQS 4.1-4.2 reinforce the concept of expectation value.

#### Notes to students

- All states appearing throughout are normalized, i.e., for a state  $|\chi\rangle = a|\chi_1\rangle + b|\chi_2\rangle$ ,  $|a|^2 + |b|^2 = 1$ .
- The observable  $S_i$  is the i-component of the spin. The corresponding operator is  $\hat{S}_i$ , for  $i = \{x, y, z\}$ .
- For instance, the observable  $S_z$  is the z component of the spin, and the corresponding operator is  $\hat{S}_z$ .
- In all instances,  $\hat{H} = C\hat{S}_z$ , where C is a suitable constant.

#### Notes for CQS 1.1-1.2

- The following 2 questions present the same concept: once with a spin operator, and once with the Hamiltonian. They address what the application of a Hermitian operator is, and what it is not.
  - Application of an operator to a state *is not* the measurement of an observable.
  - Application of an operator to a state *is* a mathematical process that transforms the state.
- Option A,  $\hat{H}|\chi\rangle = E|\chi\rangle$ , for CQS 1.2 looks like the time-independent Schrödinger equation. However,  $|\chi\rangle$  must be an eigenstate of the Hamiltonian for the equality to hold, known as “solving the energy eigenvalue problem.”
- Note that students are assumed to know that if  $\hat{H} \propto \hat{S}_z$ , then  $|E_{\pm}\rangle = |\pm z\rangle$ .

#### CQS 1.1

Which one of the following is correct if  $|\chi\rangle = a|+z\rangle + b|-z\rangle$ ?

- $\hat{S}_z|\chi\rangle = \frac{\hbar}{2}|\chi\rangle$
- $\hat{S}_z|\chi\rangle = \frac{\hbar}{2}|+z\rangle$  or  $-\frac{\hbar}{2}|-z\rangle$
- $\hat{S}_z|\chi\rangle = \frac{\hbar}{2}$  or  $-\frac{\hbar}{2}$
- $\hat{S}_z|\chi\rangle = |+z\rangle$  or  $|-z\rangle$
- None of the above

#### CQS 1.2

Which one of the following is correct regarding the Hamiltonian operator  $\hat{H}$  acting on a generic state  $|\chi\rangle = a|+z\rangle + b|-z\rangle$ ?

(Note: Here,  $\hat{H} = C\hat{S}_z$ , where C is a suitable constant and, for simplicity, the energies corresponding to the  $|+z\rangle$  and  $|-z\rangle$  states are given as  $E_+$  and  $E_-$ , respectively.)

- $\hat{H}|\chi\rangle = E|\chi\rangle$ , where E is a constant
- $\hat{H}|\chi\rangle = E_+|+z\rangle$  or  $E_-|-z\rangle$
- $\hat{H}|\chi\rangle = E_+$  or  $E_-$
- $\hat{H}|\chi\rangle = aE_+|+z\rangle + bE_-|-z\rangle$
- None of the above

#### CQS 1.3

Consider the following in the 2-D Hilbert space corresponding to electron spin:

- For every observable  $Q$ , there is a corresponding Hermitian operator  $\hat{Q}$ .
- The operator  $\hat{Q}$  has two eigenstates,  $|1\rangle$  and  $|2\rangle$ .
- The eigenstates are associated with the eigenvalues  $q_1$  and  $q_2$ , such that  $\hat{Q}|1\rangle = q_1|1\rangle$  and  $\hat{Q}|2\rangle = q_2|2\rangle$ .

Choose all of the statements that are correct about a measurement of the observable  $Q$  made in the generic state  $|\chi\rangle = a|1\rangle + b|2\rangle$ .

1. The measurement of an observable  $Q$  will collapse the state  $|\chi\rangle$  into an eigenstate of the corresponding operator  $\hat{Q}$ .
  2. A measurement of an observable  $Q$  must return one of the eigenvalues  $q$  of the Hermitian operator  $\hat{Q}$ .
  3. An operator  $\hat{Q}$  acting on state  $|\chi\rangle$  is equivalent to the measurement of the observable  $Q$ . The measurement process is given by  $\hat{Q}|\chi\rangle = q_1|1\rangle$  or  $q_2|2\rangle$
- A. 1 only    B. 1 and 2 only    C. 1 and 3 only    D. 2 and 3 only    E. All of the above

## CLASS DISCUSSION

Notes for CQS 1.3

- The preceding question addresses the incorrect notion that making a measurement of an observable in a quantum state is equivalent to having the corresponding operator act on that state.
- In addition, other concepts related to measurement can be discussed:
  - Results of a measurement can only be eigenvalues of the operator corresponding to the observable being measured
  - Probabilities are determined by the state  $|\chi\rangle$  at the time of measurement. The probabilities of obtaining  $\lambda_i$  are governed by the Born rule:  $|\langle\lambda_i|\chi\rangle|^2$  where  $|\lambda_i\rangle$  is an eigenstate with eigenvalue  $\lambda_i$  of the operator corresponding to the observable measured. The outcome of a measurement is, in general, not ensured, but can be predicted statistically.
  - The measurement collapses the state to the eigenstate associated with the eigenvalue that was measured.
- *Operators that commute:* When  $\hat{H}$  and  $\hat{S}_z$  commute,  $[\hat{H}, \hat{S}_z] = 0$  and the operators' normalized eigenvectors are identical. Eigenvalues are unique to each operator, with specified units ( $\hat{H}$  has units of energy, and  $\hat{S}_z$  has units of angular momentum).
- If Larmor precession has already been discussed, students can be reminded that the phenomenon is governed by the Hamiltonian (here  $\hat{H} = C\hat{S}_z$ , as is conventional, though of course it is equivalent if another component of spin is chosen to commute with the Hamiltonian, e.g.,  $\hat{H} = C\hat{S}_x$ ).

## CQS 2.1

Consider the state  $|\chi\rangle = \frac{\sqrt{12}}{5}|+z\rangle + \left(\frac{3}{5} - \frac{2i}{5}\right)|-z\rangle$ . When a measurement of the  $z$ -component of the spin ( $S_z$ ) is made in this state, which one of the following is true?

- A. The probability of measuring  $-\frac{\hbar}{2}$  is  $\left(\frac{3}{5} - \frac{2i}{5}\right)$ .
- B. The probability of measuring  $-\frac{\hbar}{2}$  is  $\frac{13}{25}$ .
- C. The probability of measuring  $-\frac{\hbar}{2}$  is  $\frac{9}{25}$ .
- D. The probability of measuring  $-\frac{\hbar}{2}$  is  $\left(\frac{3}{5} - \frac{2i}{5}\right)^2$ .
- E. The probability of measuring  $-\frac{\hbar}{2}$  is  $\frac{1}{5}$ .

Note for CQS 2.2: The following question may take students some time since it requires some calculation. Consider allowing 2-3 minutes.

## CQS 2.2

Consider the state  $|\chi\rangle = \sqrt{\frac{3}{10}}|+z\rangle + \sqrt{\frac{7}{10}}|-z\rangle$ . What is the probability of measuring a value of  $+\frac{\hbar}{2}$  for the  $x$ -component of the spin ( $S_x$ )?

$$|+z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle + |-x\rangle)$$

$$|-z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle - |-x\rangle)$$

- A)  $\frac{3}{10}$   
 B)  $\frac{(\sqrt{3}-\sqrt{7})^2}{20}$   
 C)  $\frac{(\sqrt{3}+\sqrt{7})^2}{20}$   
 D) A measurement of the  $x$ -component of spin cannot be performed on a state  $|\chi\rangle$  which is a superposition of eigenstates of  $\hat{S}_z$ .  
 E) None of the above

### CLASS DISCUSSION

Notes for CQS 2.2

- At the end, tell students that it requires a change of basis, and ask some students how they did it.
- Go over the change of basis. This will help prime students for the following questions.

### CQS 2.3

Consider an electron spin in the state  $|\chi\rangle = \sqrt{\frac{3}{10}}|+z\rangle + \sqrt{\frac{7}{10}}|-z\rangle$ . The  $x$  component of its spin ( $S_x$ ) was measured, and returned a value of  $+\frac{\hbar}{2}$ . What is the normalized state immediately after the measurement?

- A)  $\sqrt{\frac{3}{10}}|+x\rangle$   
 B)  $\frac{\sqrt{3}+\sqrt{7}}{\sqrt{20}}|+x\rangle$   
 C)  $\left(\frac{\sqrt{3}+\sqrt{7}}{\sqrt{20}}\right)^2|+x\rangle$   
 D)  $|+x\rangle$   
 E) Not enough information

Note for CQS 3.1-3.2: The relationships between the  $|\pm x\rangle$  and  $|\pm y\rangle$  states are not given because it is not necessary for students to actually change to the  $\{|+y\rangle, |-y\rangle\}$  basis and calculate the probabilities of measuring each outcome in order to correctly answer the questions.

### CQS 3.1

Consider the state  $|\chi\rangle = \sqrt{\frac{2}{3}}|+x\rangle + \sqrt{\frac{1}{3}}|-x\rangle$ . We first measure the observable  $S_x$ , then  $S_y$  immediately after. Choose all of the following statements that are true:

- I. For  $S_x$ ,  $P\left(\frac{\hbar}{2}\right) = \frac{2}{3}$ , and  $P\left(-\frac{\hbar}{2}\right) = \frac{1}{3}$ .

- II. For  $S_y$ ,  $P\left(\frac{\hbar}{2}\right) = \frac{2}{3}$ , and  $P\left(-\frac{\hbar}{2}\right) = \frac{1}{3}$ .
- III. For  $S_y$ ,  $P\left(\frac{\hbar}{2}\right) = \frac{1}{2}$ , and  $P\left(-\frac{\hbar}{2}\right) = \frac{1}{2}$ .
- IV. If  $S_x$  was measured to be  $\frac{\hbar}{2}$ , then for  $S_y$ ,  $P\left(\frac{\hbar}{2}\right) = 1$ ; if  $S_x$  was measured to be  $-\frac{\hbar}{2}$ , then for  $S_y$ ,  $P\left(-\frac{\hbar}{2}\right) = 1$ .
- A) I only  
 B) IV only  
 C) I and II only  
 D) I and III only  
 E) I and IV only

### CQS 3.2

Consider the state  $|\chi\rangle = \sqrt{\frac{2}{3}}|+x\rangle + \sqrt{\frac{1}{3}}|-x\rangle$ . We first measure the observable  $S_y$ , then  $S_x$  immediately after. Choose all of the following statements that are true:

- I. For  $S_y$ ,  $P\left(\frac{\hbar}{2}\right) = \frac{2}{3}$ , and  $P\left(-\frac{\hbar}{2}\right) = \frac{1}{3}$ .
- II. For  $S_x$ ,  $P\left(\frac{\hbar}{2}\right) = \frac{2}{3}$ , and  $P\left(-\frac{\hbar}{2}\right) = \frac{1}{3}$ .
- III. For  $S_x$ ,  $P\left(\frac{\hbar}{2}\right) = \frac{1}{2}$ , and  $P\left(-\frac{\hbar}{2}\right) = \frac{1}{2}$ .
- IV. If  $S_y$  was measured to be  $\frac{\hbar}{2}$ , then for  $S_x$ ,  $P\left(\frac{\hbar}{2}\right) = 1$ ; if  $S_y$  was measured to be  $-\frac{\hbar}{2}$ , then for  $S_x$ ,  $P\left(-\frac{\hbar}{2}\right) = 1$ .
- A. I only  
 B. II only  
 C. III only  
 D. II and IV  
 E. III and IV

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## CLASS DISCUSSION

Notes for CQS 3.1-3.2

- Note that in the preceding questions CQS 3.1-3.2, option IV can never be true because  $\hat{S}_x$  and  $\hat{S}_y$  do not commute.
  - CQS 3.1 invites a discussion that, once the state has collapsed to an eigenstate of one of the spin operators, measurement (in immediate succession) of the same spin component does not change the state, while measurement of either of the other two spin components will return spin-up and spin-down with equal probability.
- 

### CQS 4.1

Choose all of the correct expressions for the expectation value  $\langle S_y \rangle$  in a state  $|\chi\rangle = a|+y\rangle + b|-y\rangle$ .

- I.  $\langle \chi | \hat{S}_y | \chi \rangle$   
 II.  $|a|^2 \langle +y | \hat{S}_y | +y \rangle + |b|^2 \langle -y | \hat{S}_y | -y \rangle$

- III.  $(|a|^2 - |b|^2) \frac{\hbar}{2}$
- A. I only
  - B. III only
  - C. I and II only
  - D. I and III only
  - E. All of the above

#### CQS 4.2

Given a state  $|\chi\rangle = \frac{3}{5}|+y\rangle + \frac{4}{5}|-y\rangle$ , which of the following is the expectation value  $\langle S_y \rangle$ ?

- A.  $-\frac{7}{50} \hbar$
- B.  $-\frac{1}{10} \hbar$
- C. Either  $+\frac{\hbar}{2}$  or  $-\frac{\hbar}{2}$
- D. 0
- E. None of the above

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### CLASS DISCUSSION

Notes for CQS 4.1-4.2

- Expectation values can be introduced in several complementary ways:
  - Using Dirac notation  $\langle \chi | \hat{Q} | \chi \rangle$
  - Using matrix representation in a given basis to compute  $\langle \chi | \hat{Q} | \chi \rangle$
  - Characterizing expectation value as a weighted average of measurement outcomes on an ensemble of identically prepared systems, i.e.,  $\sum_i (\text{probability of measuring } i\text{th eigenvalue}) \times (\text{value of } i\text{th eigenvalue})$
  - Showing the equivalence of all these approaches

## Appendix D Materials for Chapter 5

### Appendix D.1 Pre-test and post-test questions

The states associated with the  $z$ -component of spin with eigenvalues  $\frac{\hbar}{2}$  and  $-\frac{\hbar}{2}$  are  $|+z\rangle$  and  $|-z\rangle$ , respectively (similar notation is used for the  $x$ - and  $y$ -components). All other notations are standard.

Additionally, students were given the following notes:

- The spin operators  $\hat{S}_z$ ,  $\hat{S}_x$ , and  $\hat{S}_y$  correspond to the observables  $S_z$ ,  $S_x$ , and  $S_y$ , respectively, which in turn correspond to the  $z$ -,  $x$ -, and  $y$ -components of a spin-1/2 particle's spin. All notations are conventional.
  - $\hat{S}_z|z\rangle = \frac{\hbar}{2}|+z\rangle$        $\hat{S}_z|-z\rangle = -\frac{\hbar}{2}|-z\rangle$ ;
  - $\hat{S}_x|x\rangle = \frac{\hbar}{2}|+x\rangle$        $\hat{S}_x|-x\rangle = -\frac{\hbar}{2}|-x\rangle$
  - $\hat{S}_y|y\rangle = \frac{\hbar}{2}|+y\rangle$        $\hat{S}_y|-y\rangle = -\frac{\hbar}{2}|-y\rangle$
  - $\hat{S}^2|\pm\alpha\rangle = \frac{3\hbar^2}{4}|\pm\alpha\rangle$  where  $\alpha = x, y, z$
- Measurement in a state in which an observable is well-defined will yield a particular eigenvalue with 100% certainty (i.e., the state is an eigenstate of the operator corresponding to the observable).

Correct answers for multiple-choice questions are bolded.

1. Consider a system in the state  $|+x\rangle$ . Which of the following observables are well-defined (i.e., they can be measured with no uncertainty) in this state?
  - I.  $S^2$
  - II.  $S_x$
  - III.  $S_z$
2. Consider a system in the state  $|+x\rangle$ . Measurements of which of the following observables would yield a well-defined value?
  - I. Energy, if the Hamiltonian is  $\hat{H} = C\hat{S}_z$  (where  $C$  is an appropriate constant)
  - II. Energy, if the Hamiltonian is  $\hat{H} = C\hat{S}_x$  (where  $C$  is an appropriate constant)
  - III. Any observable whose corresponding operator commutes with  $\hat{S}_x$
3. Consider the Hermitian operators  $\hat{A}$  and  $\hat{B}$  which correspond to observables. They are *incompatible* operators. The Hamiltonian is given by  $\hat{H} = \hat{A} + \hat{B}$ . Suppose you measure energy and obtain  $E_0$ . Choose all of the following statements that are correct immediately after the measurement of energy.
  - I.  $A$  is well-defined.

- II. B is well-defined.
- III. The state collapses to an eigenstate of  $\hat{B}$  immediately after the measurement of energy.
4. For the following states of a system, does a measurement of the observable  $S_y$  yield a value with 100% probability? If the uncertainty is non-zero, calculate it.
- $\frac{1}{\sqrt{3}}|+y\rangle + \sqrt{\frac{2}{3}}|-y\rangle$
  - $|+y\rangle$
5. Consider the Hermitian operators  $\hat{S}_x$ ,  $\hat{S}_y$ , and  $\hat{S}^2$ . Suppose you made a measurement of the observable  $S_x$  for a system in some state, and obtained the value  $-\frac{\hbar}{2}$ .
- Suppose you immediately made another measurement of the observable  $S_x$ . What are the possible values that you can measure? What is the state immediately after the measurement? Explain.
  - Suppose you instead immediately measured the observable  $S_y$  after the first measurement of  $S_x$ . What are the possible values that you can measure? What is the state immediately after the measurement? Explain.
  - Suppose after the first measurement of  $S_x$ , you measured  $S_y$  in immediate succession, and  $S_x$  once again. What are the possible values that you can measure? What is the state immediately after the measurement? Explain.
  - Suppose after the first measurement of  $S_x$ , you measured  $S^2$  in immediate succession, and  $S_x$  once again. What are the possible values that you can measure? What is the state immediately after the final measurement? Explain.
  - Suppose after the first measurement of  $S_x$ , you measured  $S_y$  in immediate succession, and then  $S^2$  immediately after that. What are the possible values that you can measure? What is the state immediately after the final measurement? Explain.
6. [In-person only] What does it mean for two observables in QM to have an uncertainty relation? Explain in your own words, and imagine your audience to be a high school student.

## Appendix D.2 Clicker questions

When the state  $|\chi\rangle$  appears in the clicker questions, it refers to a generic state.

**CQS 1.1** Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_x$ , where  $C$  is an appropriate constant. Choose all of the following statements that are correct for a system in an eigenstate of  $\hat{S}_z$ , i.e.,  $|\pm z\rangle$ .

- I. The observable  $S_x$  is well-defined.
  - II. The observable  $S_z$  is well-defined.
  - III. Energy is well-defined.
- A. I only      B. II only      C. II and III only      D. I and III only      E. None of the above

**CQS 1.2** Consider a system with a Hamiltonian  $\hat{H} = C\hat{S}_x$ , where  $C$  is an appropriate constant. Choose all of the following statements that are correct for the state  $\frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$ .

- I. The observable  $S_x$  is well-defined.
  - II. The observable  $S_z$  is well-defined.
  - III. Energy is well-defined.
- A. I only      B. II only      C. III only      D. I and III only      E. None of the above

**CQS 1.3** Consider a system with a Hamiltonian  $\hat{H} = C(\hat{S}_x + \hat{S}_z)$ , where  $C$  is an appropriate constant. Choose all of the following statements that are correct for the state  $\frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$ .

- I. Since  $[\hat{S}_x, \hat{S}_z] \neq 0$ , energy cannot be well-defined in *any* state of this system.
  - II. The observable  $S_x$  is well-defined.
  - III. The observable  $S_z$  is well-defined.
- A. I only      B. II only      C. III only      D. I and II only      E. None of the above

**Class discussion for CQS 1.1-1.3**

- When an observable is well-defined in a state, measurements of this observable will return a predictable result with 100% certainty.
- The particular state  $\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$  is an eigenstate of  $\hat{S}_x$ , but  $S_x$  would not be well-defined for arbitrary coefficients of  $a|+z\rangle + b|-z\rangle$ .
- Discuss that the Hamiltonian  $\hat{H} = C(\hat{S}_x + \hat{S}_z)$  is perfectly acceptable: not only is it diagonalizable and has its own eigenstates (which are not the same as those of either  $\hat{S}_x$  or  $\hat{S}_z$ ), but it can also be realized in an experiment, e.g., by applying a magnetic field of the form  $\vec{B} = \frac{B_0}{\sqrt{2}}(\hat{x} + \hat{z})$ .

**CQS 1.4** Choose all of the following statements that are true regarding the uncertainty principle and uncertainty of measurement in QM.

- I. The uncertainty principle refers to the inability of a measuring apparatus to be infinitely precise.

- II. The uncertainty in the measurement of an observable can be determined by making a large number of measurements in identically prepared quantum systems in state  $|\chi\rangle$ , and calculating the standard deviation of those measurements.
- III. The uncertainty in the measurement of an observable can never be zero, because the product of the uncertainties of two observables whose operators do not commute must always be  $\geq \frac{\hbar}{2}$ .
- A. I only      B. II only      C. III only      D. All of the above      E. None of the above

**Class discussion for CQS 1.4**

- The uncertainty principle describes the observation that two observables whose operators do not commute (and thus whose operators do not have a complete set of simultaneous eigenstates) can never be measured with 100% certainty in the same state, i.e., the system cannot be in an eigenstate of both operators at the same time.
- The uncertainty in the measurement of an observable must be determined by performing a large number of measurements on identically prepared systems, instead of making repeated measurements on the same system. This is because the first measurement will collapse the state of a system into an eigenstate of the operator corresponding to the observable being measured, which in general will not be the initial state (unless the initial state was an eigenstate of the operator corresponding to the observable being measured).
- Emphasize to students that the generalized uncertainty principle is  $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle \chi | [\hat{A}, \hat{B}] | \chi \rangle\right)^2$ , and that  $\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$  is only the special case for position and momentum in one dimension.

**CQS 2.1** Choose all of the following statements that are correct about uncertainties in the measurement of different components of spin, for an ensemble of identical systems in an eigenstate of  $\hat{S}_z$ .

- I.  $\sigma_{S_z}^2 = 0$   
 II.  $\sigma_{S_x}^2 \neq 0$   
 III.  $\sigma_{S_y}^2 = 0$
- A. I only      B. III only      C. I and II only      D. I and III only      E. None of the above

**CQS 2.2** Choose all of the following statements that are correct about the uncertainty in the measurement of an observable  $A$  in state  $|\chi\rangle$  (which is not an eigenstate of  $\hat{A}$ ).

- I.  $\sigma_A^2 = \langle (A - \langle A \rangle)^2 \rangle$   
 II.  $\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2$   
 III.  $\sigma_A^2 = \langle (A - \langle A \rangle)^2 \rangle$
- A. II only      B. I and II only      C. I and III only      D. II and III only      E. All of the above

**CQS 2.3** \*Consider the uncertainty  $\sigma_{S_x}$  in the observable  $S_x$  in a given quantum state  $|\chi\rangle$ . Choose all the following statements that are correct. [Please give students sufficient time for this question to perform the necessary calculations.]

Note:

$$\sigma_{S_x}^2 = \langle \hat{S}_x^2 \rangle - \langle \hat{S}_x \rangle^2 \quad \langle \hat{S}_x \rangle = \langle \chi | \hat{S}_x | \chi \rangle \quad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- I.  $\sigma_{S_x} = 0$  in the state  $|z\rangle$ .
- II.  $\sigma_{S_x} = \frac{3}{5} \left(\frac{\hbar}{2}\right)$  in the state  $\frac{1}{\sqrt{5}}|+z\rangle + \frac{2}{\sqrt{5}}|-z\rangle \doteq \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .
- III.  $\sigma_{S_x} = \frac{1}{5} \left(\frac{\hbar}{2}\right)$  in the state  $\frac{1}{\sqrt{5}}|+z\rangle + \frac{2}{\sqrt{5}}|-z\rangle \doteq \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

- A. II only    B. III only    C. I and II only    D. I and III only    E. None of the above

**CQS 2.4** \*Consider the state  $|\chi\rangle = \frac{1}{\sqrt{5}}|+z\rangle + \frac{2}{\sqrt{5}}|-z\rangle$ . You make a measurement of  $S_z$  in this state and obtain  $\frac{\hbar}{2}$ . Choose all of the following statements that are true.

Note:

$$\sigma_{S_z}^2 = \langle \hat{S}_z^2 \rangle - \langle \hat{S}_z \rangle^2 \quad \langle \hat{S}_z \rangle = \langle \chi | \hat{S}_z | \chi \rangle \quad \hat{S}_z^2 = \hat{S}_x^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- I. A measurement of  $S_x$  made in immediate succession will have uncertainty  $\sigma_{S_x} = \frac{3}{5} \left( \frac{\hbar}{2} \right)$ .  
 II. A measurement of  $S_x$  made in immediate succession will have uncertainty  $\sigma_{S_x} = \frac{\hbar}{2}$ .  
 III. A measurement of  $S_x$  made in immediate succession will have uncertainty  $\sigma_{S_x} = 0$ .

- A. I only    B. II only    C. III only    D. None of the above    E. Not enough information

\*Note: CQS 2.3-2.4 were added to the CQS for the in-person implementation to give students practice with numerically calculating measurement uncertainty.

**CQS 2.5** The generalized uncertainty principle for two operators is  $\sigma_A^2 \sigma_B^2 \geq \left( \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$ . For  $\hat{S}_x$  and  $\hat{S}_y$ , this means  $\sigma_{S_x}^2 \sigma_{S_y}^2 \geq \left( \frac{1}{2i} \langle [\hat{S}_x, \hat{S}_y] \rangle \right)^2$ . Choose all of the following statements that are correct for a system consisting of a spin-1/2 particle.

- I. If the state of the system is  $|z\rangle$ , we have  $\langle [\hat{S}_x, \hat{S}_y] \rangle = \langle +z | [\hat{S}_x, \hat{S}_y] | +z \rangle = 0$ .  
 II. If the state of the system is  $|y\rangle$ , we have  $\langle [\hat{S}_x, \hat{S}_y] \rangle = \langle +y | [\hat{S}_x, \hat{S}_y] | +y \rangle = 0$ .  
 III. If we measure the observables  $S_x$  or  $S_y$  in the state  $|+y\rangle$ , we will measure  $\frac{\hbar}{2}$  in each case with 100% certainty.

- A. II only    B. I and II only    C. II and III only    D. All of the above  
 E. None of the above

**Class discussion for CQS 2.1-2.5**

- Emphasize that the uncertainty in the measurement of an observable depends on both the observable being measured and the state of a given system.
- Note that the uncertainty in the measurement of a component of spin can in fact be zero for choice II,  $\sigma_{S_x}^2 \sigma_{S_y}^2 = \left( \frac{1}{2i} \langle +y | [\hat{S}_x, \hat{S}_y] | +y \rangle \right)^2 = 0$ , because the commutator  $[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$  and the expectation value of  $S_z$  in the state  $|+y\rangle$  is  $\langle +y | \hat{S}_z | +y \rangle = 0$ . At the same time,  $\sigma_{S_y}^2 = 0$  in an eigenstate of  $\hat{S}_y$ , and thus  $\sigma_{S_x}^2 \sigma_{S_y}^2 = \frac{1}{2i} \left( \langle +y | [\hat{S}_x, \hat{S}_y] | +y \rangle \right)^2$  reduces to  $0 = 0$ , which is true but not useful. There are other ways to show this as well. In summary, to calculate  $\langle [\hat{S}_x, \hat{S}_y] \rangle$ :
  - If the state of the system is  $|+z\rangle$ , we have  $\langle [\hat{S}_x, \hat{S}_y] \rangle = \langle +z | [\hat{S}_x, \hat{S}_y] | +z \rangle = i \frac{\hbar^2}{2}$ .
  - If the state of the system is  $|+y\rangle$ , we have  $\langle [\hat{S}_x, \hat{S}_y] \rangle = \langle +y | [\hat{S}_x, \hat{S}_y] | +y \rangle = 0$ .

**CQS 3.1** Consider the following pairs of operators, which correspond to observables. Which of these pairs of operators can have simultaneous eigenstates?

Note:  $[\hat{S}_y, \hat{S}^2] = 0$

- I.  $\hat{S}_y$  and  $\hat{S}_x$

- II.  $\hat{S}_y$  and  $(\hat{S}_x^2 + \hat{S}_z^2) = (\hat{S}^2 - \hat{S}_y^2)$   
 III.  $\hat{S}_z$  and  $(C\hat{S}_x + \hat{S}_z^2)$ , where  $C$  is an appropriate constant.  
 A. II only    B. I and II only    C. I and III only    D. II and III only    E. None of the above

**CQS 3.2** Consider the following pairs of observables. Do quantum states exist for which the measurements of both observables in a system will yield definite values with 100% certainty? Select the pairs for which the answer is yes.

- I.  $S_y$  and  $(S_x^2 + 2S_y^2 + S_z^2) = (S^2 + S_y^2)$   
 II.  $S_y$  and  $(S_x + S_y + S_z)$   
 III.  $S_y$  and  $(CS_y + S_x S_y^2)$ , where  $C$  is an appropriate constant.  
 A. I only    B. II only    C. III only    D. I and II only    E. None of the above

*Note:* In a spin-1/2 system only,  $\hat{S}_x^2 = \hat{S}_y^2 = \hat{S}_z^2 = \frac{\hbar^2}{4}\hat{I}$ . This property is not important for answering the questions CQS 3.1-3.2, which are generalizable to all spin systems. The property can, however lead to behavior such as  $[\hat{S}_x, \hat{S}_x^2 + \hat{S}_z^2] = 0$ , which is not generalizable.

**CQS 3.3** Choose all of the following statements that are correct.

- I.  $[\hat{S}_x, \hat{S}_y \hat{S}_x] = \hat{S}_x [\hat{S}_x, \hat{S}_y]$   
 II.  $[\hat{S}_x, \hat{S}_y \hat{S}_x] = \hat{S}_y [\hat{S}_x, \hat{S}_x]$   
 III.  $[\hat{S}_x, C_1 \hat{S}^2 + C_2 \hat{S}_z] = C_2 [\hat{S}_x, \hat{S}_z]$  (where  $C_1$  and  $C_2$  are non-zero constants)  
 A. I only    B. II only    C. III only    D. I and III only    E. II and III only

**CQS 3.4** \*\*The operators  $\hat{S}_x$  and  $\hat{S}^2$  commute. Suppose you measured  $S_x$  in an initial state, in a spin-1/2 system, and obtained a value of  $\frac{\hbar}{2}$ . What will a subsequent measurement of  $S^2$  yield?

- A.  $\frac{\hbar}{2}$     B. Either  $\frac{\hbar^2}{4}$  or  $-\frac{\hbar^2}{4}$     C.  $\frac{\hbar^2}{4}$     D.  $\frac{3\hbar^2}{4}$     E. Either  $\frac{3\hbar^2}{4}$  or  $-\frac{3\hbar^2}{4}$

**CQS 3.5** \*\*The operators  $\hat{S}_x$  and  $\hat{S}^2$  commute. Suppose you measure  $S_x$  in some initial state, for a spin-1/2 system, and obtain a value of  $\frac{\hbar}{2}$ . What will the state be if  $S^2$  is measured in immediate succession?

- A.  $|+x\rangle$   
 B. Either  $|+x\rangle$  or  $|-x\rangle$   
 C. The system can be in any of the eigenstates of  $\hat{S}_x$ ,  $\hat{S}_y$ , or  $\hat{S}_z$   
 D. The system can be in any state in the Hilbert space after the measurement, since every state is an eigenstate of  $\hat{S}^2$ .  
 E. Not enough information

*\*\*Note: CQS 3.4-3.5 were added to the CQS for the in-person implementation after observing the student difficulties present after the online implementation.*

**Class discussion for CQS 3.1-3.5**

- Discuss how the operator  $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$  commutes with all three operators  $\hat{S}_x$ ,  $\hat{S}_y$ , and  $\hat{S}_z$ .

- Note that neither  $\hat{S}_x$  nor  $\hat{S}_y$  would commute with an operator  $\hat{S}_x^2 + \hat{S}_y^2$ . However,  $\hat{S}_x$  would commute with  $\hat{S}_y^2 + \hat{S}_z^2$ , which can be alternatively expressed as  $\hat{S}^2 - \hat{S}_x^2$ . An analogous case holds for  $\hat{S}_y$ .
- The situation is special for  $\hat{S}^2$  for spin-1/2 systems, because  $\hat{S}_x^2$ ,  $\hat{S}_y^2$ , and  $\hat{S}_z^2$  are all proportional to the identity operator  $\hat{I}$ , such that, e.g.,  $\hat{S}_x$  does commute with  $\hat{S}_x^2 + \hat{S}_y^2$ .
- Discuss the connection between a complete set of simultaneous eigenstates of commuting operators and the implications for the measurements of commuting observables in immediate succession. (Students may need help seeing the connection between CQS 3.1-3.2.)
  - Emphasize to students the implications of degeneracy in the eigenvalue spectrum of  $\hat{S}^2$ , i.e., every eigenstate of  $\hat{S}_x$  is an eigenstate of  $\hat{S}^2$ , but not the other way around.
- Emphasize also that  $\hat{S}_x$ ,  $\hat{S}_y$ , and  $\hat{S}_z$  are on the same footing with regard to commutation relations to one another and to  $\hat{S}^2$ .
  - It is conventional to write  $\hat{S}_x, \hat{S}_y, \hat{S}_z$  for a spin-1/2 system (which are proportional to the Pauli matrices) in a basis consisting of eigenstates of  $\hat{S}_z$  (but this is like defining the x-axis in classical mechanics to be horizontal).
  - Any one of these, for example,  $\hat{S}_z$ , is “special” for a system only if the Hamiltonian of the system commutes with  $\hat{S}_z$ .

**CQS 4.1** \*\*\*Consider the Hermitian operators  $\hat{A}$  and  $\hat{B}$ , which correspond to observables  $A$  and  $B$ , respectively. They are *compatible* operators ( $[\hat{A}, \hat{B}] = 0$ ). The Hamiltonian is given by  $\hat{H} = \hat{A} + \hat{B}$ . All three operators have non-degenerate eigenstates. Suppose you measure the observable  $A$  and obtain  $A_0$ . After this measurement, the state is definitely an eigenstate of which of the following operators?

- I.  $\hat{A}$
  - II.  $\hat{B}$
  - III.  $\hat{H}$
- A. I only    B. I and II only    C. I and III only    D. All of the above    E. None of the above

**CQS 4.2** \*\*\*Consider the Hermitian operators  $\hat{C}$  and  $\hat{D}$ , which correspond to observables. They are *incompatible* operators ( $[\hat{C}, \hat{D}] \neq 0$ ). The Hamiltonian is given by  $\hat{H} = \hat{C} + \hat{D}$ . Suppose you measure  $C$  and obtain  $C_0$ . After this measurement, the state is definitely an eigenstate of which of the following operators?

- I.  $\hat{C}$
  - II.  $\hat{D}$
  - III.  $\hat{H}$
- A. I only    B. I and II only    C. I and III only    D. All of the above    E. None of the above

\*\*\*Note: For the online implementation, the choices for CQS 4.1-4.2 read as follows. (The operators in the preceding question CQS 4.2 were renamed  $\hat{C}$  and  $\hat{D}$  for the in-person implementation to avoid confusion.)

Consider the Hermitian operators  $\hat{A}$  and  $\hat{B}$ , which correspond to observables. They are compatible [CQS 4.1] / incompatible [CQS 4.2] operators. The Hamiltonian is given by  $\hat{H} = \hat{A} + \hat{B}$ . Suppose you measure energy and obtain  $E_0$ . Choose all of the following that are correct after the measurement of energy:

- I.  $A$  is well-defined.
  - II.  $B$  is well-defined.
  - III. The state collapses to an eigenstate of  $\hat{B}$  immediately after the measurement of energy.
- CQS 4.1: A. I only    B. III only    C. I and II only    D. II and III only    E. All of the above  
 CQS 4.2: A. I only    B. III only    C. I and II only    D. II and III only    E. None of the above

**CQS 4.3** Choose all of the following statements that are correct if  $\hat{A}$  and  $\hat{B}$  are *incompatible* operators with non-degenerate eigenstates.

- I. It is impossible to find a complete set of simultaneous eigenstates for  $\hat{A}$  and  $\hat{B}$ .
  - II. In a given quantum state, for three successive measurements  $A \rightarrow B \rightarrow A$  (*assuming no time evolution of the state has taken place*), the two measurements of  $A$  must yield the same value.
  - III. It is possible to infer the value of the observable  $B$  after the measurement of the observable  $A$  returns a particular value for  $A$ .
- A. I only    B. III only    C. I and II only    D. All of the above    E. None of the above

## Appendix E Materials for Chapter 6

### Appendix E.1 Tutorial on the Bloch sphere

#### Bloch sphere

This tutorial on the Bloch sphere deals only with two-state systems, such as a single qubit. Additionally, the tutorial deals only with pure states, and not mixed states.

When a qubit is measured in a particular basis, it yields a particular eigenvalue corresponding to specific basis state to which the state collapsed. Thus, because of the state collapse, we know the eigenvalue and the state after the measurement. In this tutorial, the term “measurement outcome” will generally refer to the collapsed state after the measurement.

To maximize the utility of this tutorial during class time, the following warm-up section can be assigned ahead of time. Many of these exercises are straightforward but involve somewhat lengthy algebraic manipulations that students may need little external help with, and they lay a foundation for the following sections which contain questions that would benefit from class discussion.

#### Warm-up: Construction

*You should be able to:*

- Describe the connection between a two-dimensional quantum state written in Dirac notation and its representation on the Bloch sphere
- Identify where the states  $|0\rangle$ ,  $|1\rangle$ ,  $|\pm\rangle$ ,  $|\pm i\rangle$  are on the Bloch sphere
- Identify that multiplying by an overall phase does not change the state, but modifying a relative phase between the basis states does

#### **Reading the Bloch sphere:**

In the diagrams used in this tutorial, the sphere is oriented so that both the x- and y-axes appear offset from the center by equal amounts. For practical reasons, the positive z-axis is also slightly tilted toward the viewer. A beach ball is illustrated below to get the point across.

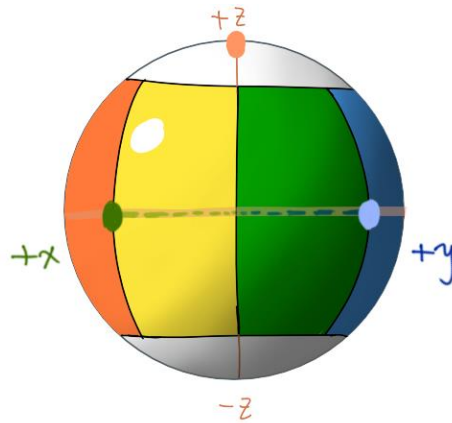


Figure 1: With the  $z$ -axis placed straight up and down (vertically), the  $x$ - $y$  plane, viewed edge-on, is completely flat and nearly invisible!

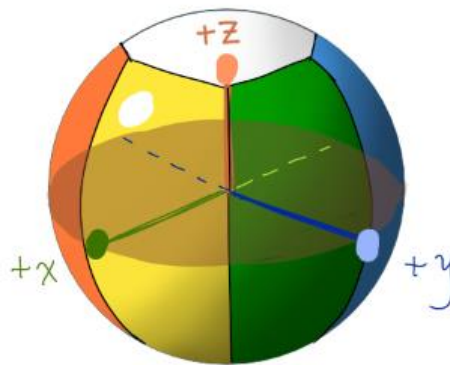


Figure 2: The top half of the ball is angled slightly toward the viewer. (Notice that the  $z$ -,  $x$ -, and  $y$ -axes do reach the edge of the ball, even though it does not look that way in this perspective.) This is the view from which all the following Bloch sphere diagrams are drawn.

### Orientation

#### What is the Bloch sphere?

The Bloch sphere is a way of visually representing states of a two-state system, such as a qubit. When the given state and the measurement basis are plotted on the Bloch sphere, one can relatively easily get a sense of the outcomes of a measurement and their probabilities.

The Bloch sphere is a spherical surface of radius 1, with polar angle  $\theta$  and azimuthal angle  $\phi$  that are the same as the angles used in spherical coordinates (as conventionally defined in physics). That is,  $\theta$  begins from the positive  $z$ -axis and sweeps toward the equator. Meanwhile, when looking down the positive  $z$ -axis,  $\phi$  begins from the positive  $x$ -axis and sweeps counterclockwise about the  $z$ -axis. Any point on the surface of the Bloch sphere represents a valid pure quantum state.

#### I've chosen a point on the Bloch sphere... what does it mean?

Any point on the Bloch sphere can be specified with a unique  $\theta$  and  $\phi$ . Each state  $|q\rangle$  that can be plotted on the Bloch sphere is represented in the  $\{|0\rangle, |1\rangle\}$  basis as  $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ . (When plotted in two-dimensional Hilbert space with complex coefficients, the angle that the state forms with the  $|0\rangle$  axis is  $\frac{\theta}{2}$ .) We have the freedom to choose an overall phase such that the  $|0\rangle$  component is real and non-negative, and we usually restrict the angles such that  $0 \leq \theta \leq \pi$  and  $0 \leq \phi < 2\pi$ , to ensure a one-to-one mapping.

### What does an orthonormal basis look like on the Bloch sphere?

For any qubit, each measurement has a unique basis associated with it, which includes two orthonormal basis states, one for each possible outcome. To obtain an orthonormal basis on the Bloch sphere, simply locate the two ends of any possible diameter of the sphere. For instance, if the  $|0\rangle$  state points straight up, then the  $|1\rangle$  state points straight down. Likewise, states that point directly left and right (or any two states on “opposite” points of the Bloch sphere) are also orthonormal to each other and can be used as measurement basis states. What’s more, because of the complete spherical symmetry, we are free to label any pair of such basis states as the standard basis  $\{|0\rangle, |1\rangle\}$ , and thus define the  $z$ -axis in any direction we would like to choose.

2. Verify that a complex number in the form  $e^{i\phi} = \cos\phi + i\sin\phi$  has unit modulus (i.e., magnitude of 1) by calculating  $|e^{i\phi}|^2$ . Is the state  $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$  normalized for all values of  $\theta$  and  $\phi$ ?
3. Consider four qubits in the following states:
  - $|q_{A1}\rangle = a|0\rangle + b|1\rangle$
  - $|q_{A2}\rangle = -a|0\rangle - b|1\rangle$
  - $|q_{B1}\rangle = a|0\rangle - b|1\rangle$
  - $|q_{B2}\rangle = e^{i\phi}a|0\rangle - e^{i\phi}b|1\rangle$
  - (A) List the outcomes, and calculate and compare the probabilities of measuring those outcomes, when the states  $|q_{A1}\rangle$  and  $|q_{A2}\rangle$  are measured in the  $\{|0\rangle, |1\rangle\}$  basis.
  - (B) List the outcomes, and calculate and compare the probabilities of measuring these outcomes, when the states  $|q_{B1}\rangle$  and  $|q_{B2}\rangle$  are measured in the  $\{|0\rangle, |1\rangle\}$  basis.
  - (C) Is the probability of measuring  $|0\rangle$  or  $|1\rangle$  any different between parts (A) and (B)?
  - (D) The  $\{|+\rangle, |-\rangle\}$  basis consists of the states  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$  when expressed in the  $\{|0\rangle, |1\rangle\}$  basis.

List the outcomes, and calculate and compare the probabilities of measuring those outcomes, when the states  $|q_{A1}\rangle$  and  $|q_{A2}\rangle$  are measured in the  $\{|+\rangle, |-\rangle\}$  basis.

Hint:  $|q\rangle = a|0\rangle + b|1\rangle$  is expressed in the  $\{|+\rangle, |-\rangle\}$  basis as  $|q\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$ .

- (E) List the outcomes, and calculate and compare the probabilities of measuring these outcomes, when the states  $|q_{B1}\rangle$  and  $|q_{B2}\rangle$  are measured in the  $\{|+\rangle, |-\rangle\}$  basis.
- (F) Is the probability of measuring  $|+\rangle$  or  $|-\rangle$  any different between parts (D) and (E)?
- (G) Is it possible to tell the states  $|q_{A1}\rangle$ ,  $|q_{A2}\rangle$ ,  $|q_{B1}\rangle$ , or  $|q_{B2}\rangle$  apart from one another in the  $\{|0\rangle, |1\rangle\}$  basis? In the  $\{|+\rangle, |-\rangle\}$  basis?

*Checkpoint*

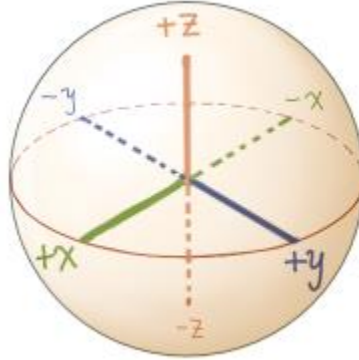
- For the state of a qubit (e.g.,  $|q\rangle = a|0\rangle + b|1\rangle$  in the  $\{|0\rangle, |1\rangle\}$  basis), multiplying  $a$  and  $b$  by the same phase  $e^{i\phi}$  (an “overall phase”)  $|q\rangle$  represents an equivalent state. In contrast, multiplying only one of  $a$  or  $b$  by a phase (thus changing their “relative phase”) will change the state.
- When states differ by a relative phase in a particular basis, measurement in that basis is unable to distinguish between them. However, transforming to any other basis will allow one to tell such states apart.
- Note that  $|q_{A2}\rangle = -a|0\rangle - b|1\rangle$  is a special case of  $e^{i\phi}a|0\rangle + e^{i\phi}b|1\rangle$  where  $e^{i\phi} = -1$  and  $\phi = \pi$ . Other simple values of  $e^{i\phi}$  include  $e^{i\frac{\pi}{2}} = i$  and  $e^{i\frac{3\pi}{2}} = -i$ .

4. A state  $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$  on the Bloch sphere (which is a unit sphere) can be visualized as a unit vector  $\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$  in spherical coordinate space, with Cartesian coordinates  $x = \sin\theta\cos\phi$ ,  $y = \sin\theta\sin\phi$ , and  $z = \cos\theta$ . (Notice that here,  $\theta$  is no longer divided by 2; recall that  $0 \leq \theta \leq \pi$  and  $0 \leq \phi < 2\pi$ . We will explain later why  $\theta$  goes to  $\frac{\theta}{2}$ , while  $\phi$  remains unchanged.)

Find the spherical coordinate vectors (known in the context of the Bloch sphere as the “Bloch vectors”) that correspond to the following states. Fill out the table to make the process easier.

State	Value of $\theta$ ?	Value of $\phi$ ?	$\hat{\mathbf{r}} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$
$ 0\rangle = 1 0\rangle + 0 1\rangle$			
$ 1\rangle$			
$ +\rangle = \frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 1\rangle$			
$ -\rangle = \frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 1\rangle$			
$ +i\rangle = \frac{1}{\sqrt{2}} 0\rangle + \frac{i}{\sqrt{2}} 1\rangle$			
$  -i\rangle = \frac{1}{\sqrt{2}} 0\rangle - \frac{i}{\sqrt{2}} 1\rangle$			

Draw and label each of the above states on the Bloch sphere below. Treat the Bloch vector as a spherical coordinate vector with radius 1, polar angle  $\theta$  defined with respect to from the positive  $z$ -axis, and azimuthal angle  $\phi$  defined with respect to from the positive  $x$ -axis.



Check your answers with the simulation on the following site: [https://www.st-andrews.ac.uk/physics/quvis/simulations\\_html5/sims/blochsphere/blochsphere.html](https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/blochsphere/blochsphere.html)

For each state in the left-hand column, insert your values into the simulation for  $\theta$  and  $\phi$  and verify that the Bloch vector is where you said it was.

5. Notice that a state written in the form  $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$  has a real and non-negative  $|0\rangle$  component. If the  $|0\rangle$  component of a state is negative, complex or imaginary, we can still plot it on the Bloch sphere, but we must first multiply it by an overall phase (a factor of  $e^{i\psi}$  for some angle  $\psi$ ) to find the values of  $\theta$  and  $\phi$ .

For each of the following states, multiply by an overall phase  $e^{i\psi}$  that makes the  $|0\rangle$  component real and positive. Write the resulting state. (Note: you must explicitly find  $e^{i\psi}$ , but you do not need to find  $\psi$  itself.)

$$(A) |a\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$(B) |b\rangle = \frac{i}{\sqrt{3}}|0\rangle + \frac{1+i}{\sqrt{3}}|1\rangle$$

6. Starting from the state  $|q\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1-i}{\sqrt{3}}|1\rangle$ , find the values of  $\theta$  and  $\phi$  that express  $|q\rangle$  in the form  $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ . Again,  $0 \leq \theta \leq \pi$  and  $0 \leq \phi < 2\pi$ .

Hint: Use the amplitude of  $|0\rangle$  to find the value of  $\frac{\theta}{2}$  first. Then, using this angle, factor out  $\sin\frac{\theta}{2}$  from the complex amplitude of  $|1\rangle$  and express the remaining complex number in the form  $e^{i\phi} = \cos\phi + i\sin\phi$  (i.e., find the angle  $\phi$ ). You are guaranteed to find a unique answer because the states are normalized.

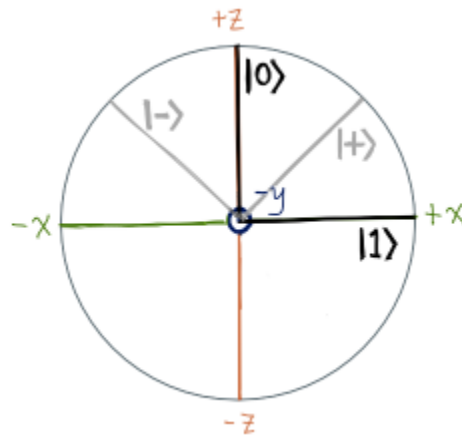
Note that in these expressions,  $\theta$  is divided by 2! The angle made in Hilbert space (in which coefficients are generally complex) between the state and the  $|0\rangle$  axis is exactly half of the angle  $\theta$  made on the Bloch sphere between the corresponding Bloch vector and the  $|0\rangle$  axis.

*Checkpoint*

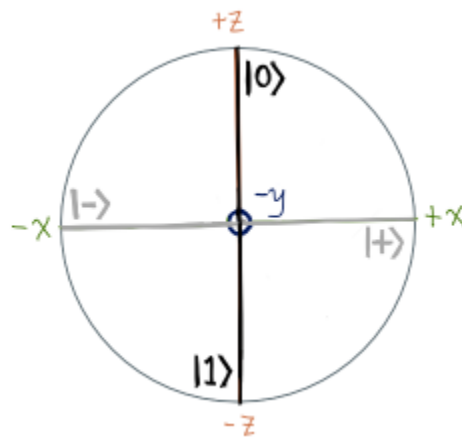
- If you are having trouble reasoning whether you should multiply or divide by the factor of 2, a simple way to remember is that on the Bloch sphere, angles are always *bigger*, while the same angle in Hilbert space is *smaller*. (Remember: Bloch is bigger.)
- When  $\theta = 0$  or  $\pi$ ,  $\phi$  can actually take any value. Does this make sense?

Consider the following conversation between two students.

**Student 1:** Orthonormal basis states, like  $|0\rangle$  and  $|1\rangle$ , are oriented at right angles ( $90^\circ$ ) from each other on the Bloch sphere, like in the diagram below. The same is true for  $|+\rangle$  and  $|-\rangle$ .



**Student 2:** No, I thought orthonormal basis states were found opposite each other on the Bloch sphere. So if  $|0\rangle$  is placed on the north pole, then  $|1\rangle$  would be on the south pole.  $|+\rangle$  and  $|-\rangle$  would also be on opposite poles, as seen below.



Which student(s), if any, do you agree with? Explain.

*Checkpoint*

- On the Bloch sphere, orthonormal basis states are found exactly opposite each other, or equivalently, they are spaced  $180^\circ$  apart. They are *not* found  $90^\circ$  apart.

Consider the following conversation between several students.

**Student 1:** The  $x$ -,  $y$ -, and  $z$ -axes are orthogonal to each other on the Bloch sphere, which is a sphere with unit radius. The states  $|0\rangle$  and  $|+\rangle$  have unit length and they lie at  $90^\circ$  from each other, so they form an orthonormal basis.

**Student 2:** We can express any point on the Bloch sphere using spherical coordinates  $r$ ,  $\theta$ , and  $\phi$ , which can be converted to Cartesian coordinates  $x$ ,  $y$ , and  $z$ . I wonder what a state would look like if we decided to express it in the  $\{|0\rangle, |+\rangle\}$  basis, which would be like making a measurement in the  $x$ - $z$  plane. You would take projective measurements along either the  $|0\rangle$  state or the  $|+\rangle$  state.

**Student 3:** Slow down. I thought an orthonormal basis for a qubit meant that taking the inner product of a basis state with itself like  $\langle 0|0\rangle$  gives you 1, and taking the inner product with the other basis state like  $\langle 0|1\rangle$  would give you 0. But if you take the inner product of  $|0\rangle$  with  $|+\rangle$ , you don't get 0 *or* 1!

**Student 4:** Right. It isn't very useful to express the states of a qubit in a "basis" consisting of  $|0\rangle$  and  $|+\rangle$ . You could do it, but these states don't form an orthonormal basis, and they would be much more difficult to work with as a result. This would be similar to choosing your  $x$ - and  $y$ -axes in introductory physics to be at, for example,  $45^\circ$  instead of  $90^\circ$  apart.

**Student 3:** It also doesn't make sense to make a measurement in the  $x$ - $z$  plane.

**Student 4:** Right, the Bloch sphere itself doesn't show amplitudes associated with projective measurements in an intuitive way, because  $\theta$  is twice as big and needs to be divided by 2.

Which student(s), if any, do you agree with? Explain.

*Checkpoint*

- Pairs of states found  $90^\circ$  apart from each other on the Bloch sphere, such as  $|0\rangle$ ,  $|+\rangle$ , and  $|+i\rangle$  (which correspond to the  $+z$ ,  $+x$ , and  $+y$ -axes respectively), do ***not*** comprise an orthonormal basis. Pairs of such states could still be used as a basis, but the basis would not be orthonormal.
- Put another way, the Bloch sphere is situated in a 3-D space which is ***not*** Hilbert space.

Consider the following conversation between several students.

**Student 1:** But why would a basis consisting of  $|0\rangle$  and  $|+\rangle$  not be considered an orthonormal basis, since the states are orthogonal and normalized?

**Student 2:** They're not actually orthogonal, are they? If they were, then  $\langle 0|+\rangle = 0$  would be true, but  $\langle 0|+\rangle$  actually equals  $\frac{1}{\sqrt{2}}$ .

**Student 3:** Right. You shouldn't get the "orthogonal" basis states of the qubit confused with the 90-degree angles on the Bloch sphere representation that correspond to the  $x$ -,  $y$ -, and  $z$ -axes. Remember, orthonormal basis states are actually found on opposite ends of the Bloch sphere, because the angle  $\theta$  between them appears twice as big on the Bloch sphere.  $|0\rangle$  and  $|+\rangle$ , which are  $\Delta\theta = 90^\circ$  apart on the Bloch sphere, actually have an angle  $\frac{\Delta\theta}{2} = 45^\circ$  between them when viewed in Hilbert space.

**Student 1:** Oh, I see! So for any state on the Bloch sphere, to find the state that is orthonormal to it, you can simply go to the point directly on the other side of the sphere, i.e., diametrically opposite, and you are guaranteed to have orthonormal basis states!

**Student 2:** And this way, each state will only ever have one other state on the Bloch sphere that's orthonormal to it, so the orthonormal basis will be unique!

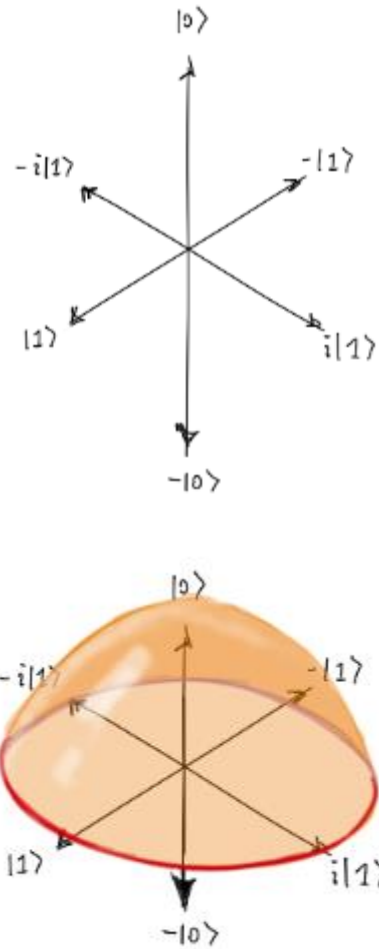
Which student(s), if any, do you agree with? Explain.

#### *Checkpoints (collected)*

- If you are having trouble reasoning whether you should multiply or divide by the factor of 2, a simple way to remember is that on the Bloch sphere, angles are always *bigger*, while the same angle in Hilbert space is *smaller*. (Remember: Bloch is bigger.)
- When  $\theta = 0$  or  $\pi$ ,  $\phi$  can actually take any value.
- On the Bloch sphere, orthonormal basis states are found exactly opposite each other, or equivalently, they are spaced  $\Delta 180^\circ$  apart. They are *not* found  $90^\circ$  apart.
- Pairs of states found  $90^\circ$  apart from each other on the Bloch sphere, such as  $|0\rangle$ ,  $|+\rangle$ , and  $|+i\rangle$  (which correspond to the  $+z$ ,  $+x$ , and  $+y$ -axes respectively), do ***not*** comprise an orthonormal basis. Pairs of such states could still be used as a basis, but the basis would not be orthonormal. The Bloch sphere is situated in a 3-D space which is ***not*** Hilbert space.

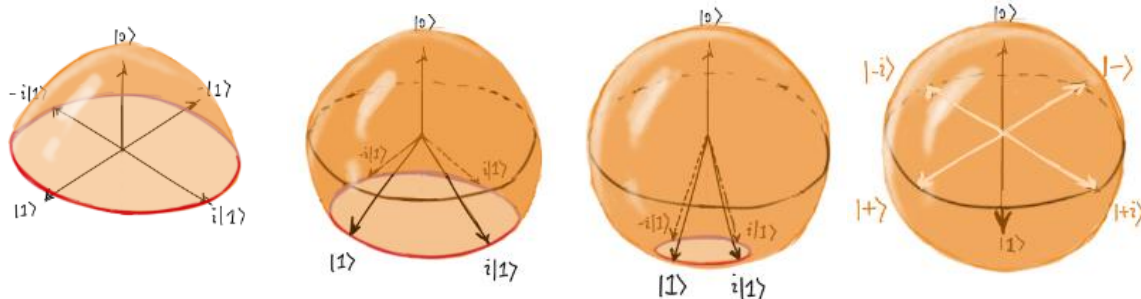
#### **How do you go from Hilbert space to the Bloch sphere?**

First we set the positive and negative  $|0\rangle$  states on the  $z$ -axis, and we can set a complex plane of all possible complex components of  $|1\rangle$  on the  $x$ - and  $y$ -axes. If we choose the overall phase so that we only consider states with a real and positive  $|0\rangle$  component, then the  $|0\rangle$  component needs neither its own complex plane nor the entire negative half. Since all states have unit length, the set of all possible states then forms a hemispherical surface as shown below.



Because the four states  $|1\rangle$ ,  $-|1\rangle$ ,  $i|1\rangle$ , and  $-i|1\rangle$  differ by only an overall phase, they all represent the same physical state. In fact, this is true of *every* state on the circular edge of the hemispherical surface (shown in red), and these are also the *only* states on the hemisphere that differ only by an overall phase rather than a relative phase. This is because the value of the  $|0\rangle$  component is zero at the equator.

Imagine fashioning this hemisphere out of some elastic, rubbery material. We then simply stretch that circular edge along a full unit sphere to meet at the south pole (the negative  $z$ -axis). We have effectively turned a half sphere into a full sphere by doubling the apparent value of  $\theta$  everywhere, so to recover the original state, we divide the new  $\theta$  by 2.

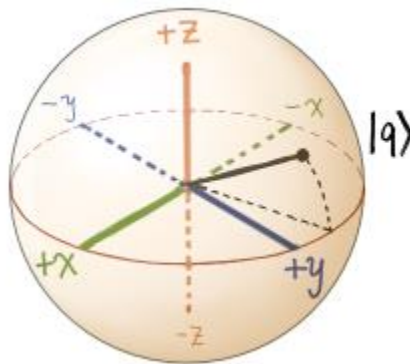


The result is the Bloch sphere. Note that there are no states on the surface that differ by an overall phase (since the unit circle of such states has now been merged into a single point). Any state in 2-D Hilbert space can be multiplied by an overall phase to yield a point on the Bloch sphere.

7. How many possible pure quantum states can be represented on the Bloch sphere?

Remember that a pure state is represented on the surface of a Bloch sphere by a point, and that each state on the Bloch sphere has a corresponding (but not necessarily unique) state in two-dimensional Hilbert space.

8. Consider a qubit (illustrated below, along with the angles that it makes with respect to the  $x$ -axis and the  $x$ - $y$  plane) in the state  $|q\rangle = a|0\rangle + b|1\rangle$ , with  $a = 0.891$  and  $b = -0.117 + 0.438i$ .



(A) When measured in the  $\{|0\rangle, |1\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|0\rangle$ ? (Use a calculator if you can; the one on any smartphone should suffice.)

(B) When measured in the  $\{|0\rangle, |1\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|1\rangle$ ?

Check your answers above with the simulation again. Use  $\theta = \frac{3\pi}{10}$  and  $\phi = \frac{7\pi}{12}$  to represent the state  $|q\rangle$ . Click on the “Show theoretical measurement outcome probabilities” box to see them.

[https://www.st-andrews.ac.uk/physics/quvis/simulations\\_html5/sims/blochsphere/blochsphere.html](https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/blochsphere/blochsphere.html)

Use this relationship for (C) and (D): If  $|q\rangle = a|0\rangle + b|1\rangle$ , then  $|q\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$ .

(C) When measured in the  $\{|+\rangle, |-\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|+\rangle$ ? (Find numerical answers by replacing  $a$  and  $b$  with the given values.)

(D) When measured in the  $\{|+\rangle, |-\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|-\rangle$ ?

Use this relationship for (E) and (F): If  $|q\rangle = a|0\rangle + b|1\rangle$ , then  $|q\rangle = \frac{a-ib}{\sqrt{2}}|+i\rangle + \frac{a+ib}{\sqrt{2}}|-i\rangle$ .

(E) When measured in the  $\{|+i\rangle, |-i\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|+i\rangle$ ?

(F) When measured in the  $\{|+i\rangle, |-i\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|-i\rangle$ ?

9. Now consider the qubit when written in the form  $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ .

(A) Calculate again the answer to part (C) of **question 8**, substituting with  $a = \cos\frac{\theta}{2}$  and  $b = \sin\frac{\theta}{2}e^{i\phi}$  instead of their numerical values.

(B) Use  $\theta = \frac{3\pi}{10}$  and  $\phi = \frac{7\pi}{12}$  to calculate numerical answers using your formula from part (A). Does your answer match with your **answer to 8(C)**?

You may find these identities helpful:

$$\frac{e^{i\phi} + e^{-i\phi}}{2} = \cos \phi$$

$$2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \sin \theta$$

## Section 1: Measurements

You should be able to:

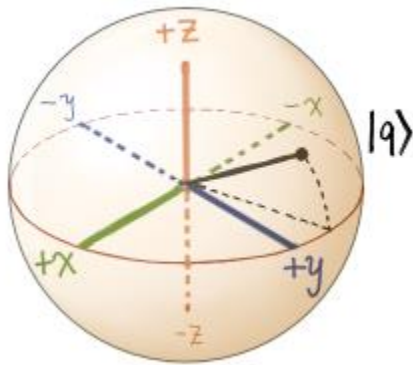
- Identify the outcomes of measurement when presented with a generic state  $|q\rangle$  on the Bloch sphere with  $\{|0\rangle, |1\rangle\}$  being the measurement basis
- Calculate the probabilities of the measurement outcomes for  $|q\rangle$  still with  $\{|0\rangle, |1\rangle\}$  being the measurement basis, using both Dirac notation (expansion coefficients of the state written as a superposition of eigenstates of an operator corresponding to the observable being measured) and the representation on the Bloch sphere ( $\cos^2 \frac{\theta}{2}$  and  $\sin^2 \frac{\theta}{2}$ )
- Describe the measurement outcomes and calculate the probabilities of those outcomes for the states  $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |+i\rangle, |-i\rangle$  when  $\{|0\rangle, |1\rangle\}, \{|+\rangle, |-\rangle\}, \{|+i\rangle, |-i\rangle\}$  are each used as measurement bases, using a method of changing basis
- Describe the measurement outcomes and calculate the probabilities of those outcomes for the given generic state  $|q\rangle$  when  $\{|0\rangle, |1\rangle\}, \{|+\rangle, |-\rangle\}, \{|+i\rangle, |-i\rangle\}$  are each used as measurement bases (by changing basis)
- Describe that any vector  $|q\rangle$  on the Bloch sphere can be used with the vector that resides on the opposite pole of the Bloch sphere  $|-q\rangle$  as an orthonormal basis
- Describe the outcomes of measurement and the probabilities of measuring those outcomes when presented with the angle between an arbitrary state  $|p\rangle$  and arbitrary measurement basis  $\{|q\rangle, |-q\rangle\}$  on the Bloch sphere

Orientation

### Measurements as seen on the Bloch sphere

When a qubit is measured, the Bloch sphere offers a straightforward geometric interpretation regarding the probability of each possible outcome. First, a measurement basis is specified as usual, and the two basis states are located on the Bloch sphere: they will always be directly opposite each other. Then, once the state to be measured is plotted on the Bloch sphere, generally speaking, the basis state to which it leans closer is the state into which it has a higher probability of collapsing when measured.

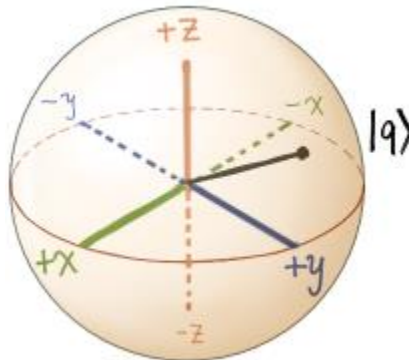
10. Consider a qubit in the state  $|q\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle$ . All measurements are made in the  $\{|0\rangle, |1\rangle\}$  basis. Express your answers to (A) and (B) in terms of  $\theta$  and  $\phi$ .
- (A) When measured in the  $\{|0\rangle, |1\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|0\rangle$ ?
- (B) What is the probability the measurement in part (A) will collapse  $|q\rangle$  into the state  $|1\rangle$ ?
- (C) A state  $|q\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle$  is drawn on the Bloch sphere below. Label the angles  $\theta$  and  $\phi$ .



Use the simulation below to answer questions 10(D) and 10(E). Use  $\theta = \frac{3\pi}{10}$  and  $\phi = \frac{7\pi}{12}$  to represent the state  $|q\rangle$ . Click on the “Show theoretical measurement outcome probabilities” box to see them.

[https://www.st-andrews.ac.uk/physics/quvis/simulations\\_html5/sims/blochsphere/blochsphere.html](https://www.st-andrews.ac.uk/physics/quvis/simulations_html5/sims/blochsphere/blochsphere.html)

- (D) Starting in this state, change the angle  $\phi$  while holding the angle  $\theta$  fixed. Describe qualitatively what happens to the state as  $\phi$  changes. How are the probabilities of measuring each outcome affected when a measurement in the  $\{|0\rangle, |1\rangle\}$  basis is made?
- (E) Starting in this state, change the angle  $\theta$  while holding the angle  $\phi$  fixed. Describe qualitatively what happens to the state as  $\theta$  changes. How are the probabilities of measuring each outcome affected when a measurement in the  $\{|0\rangle, |1\rangle\}$  basis is made?
- (F) Again the state is drawn on the Bloch sphere, but this time the polar angle  $\theta'$  is determined by starting from the  $|1\rangle$  state (i.e., starting from the negative z-axis). Label  $\theta'$  and express your answers to the following questions in terms of  $\theta'$ .



- (G) When measured in the  $\{|0\rangle, |1\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|1\rangle$ ?
- (H) What is the probability the measurement in part (G) will collapse  $|q\rangle$  into the state  $|0\rangle$ ?
- (I) Compare your answers to parts (A) and (H), and your answers to parts (B) and (G). Are your answers the same or different, and why?

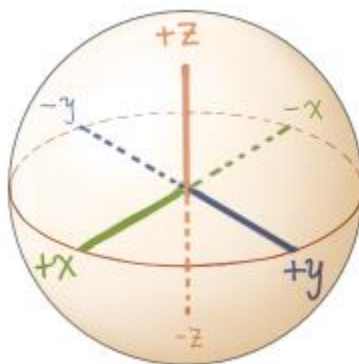
Hint: Remember that  $\sin(\frac{\pi}{2} - \psi) = \cos \psi$  and  $\cos(\frac{\pi}{2} - \psi) = \sin \psi$ .

**Checkpoint**

- On the Bloch sphere, when  $\theta$  is the angle between the given state and a “target” basis state (not necessarily the closest basis state), the probability that the measurement will yield the target state is  $\cos^2 \frac{\theta}{2}$ . The probability that the measurement will yield the state opposite the target state is  $\sin^2 \frac{\theta}{2}$ .
- For instance, if the state  $|q\rangle$  is measured in the  $\{|0\rangle, |1\rangle\}$  basis with the target state considered to be  $|0\rangle$ , then  $\theta$  is the angle between  $|q\rangle$  and  $|0\rangle$ . The probability that the state will collapse into  $|0\rangle$  is  $\cos^2 \frac{\theta}{2}$  and the probability that the state will collapse into  $|1\rangle$  is  $\sin^2 \frac{\theta}{2}$ .
- If the target state is  $|1\rangle$ , and  $\theta'$  is the angle between  $|q\rangle$  and  $|1\rangle$ , then the probability that the state will collapse into  $|1\rangle$  is  $\cos^2 \frac{\theta'}{2}$ , and the probability that the state will collapse into  $|0\rangle$  is  $\sin^2 \frac{\theta'}{2}$ .

For **question 11**, you may find the following relations helpful, and a Bloch sphere is provided for reference (you can use your answers to question 4 to figure out the placement of the six basis states in the left column):

Starting basis $\{ 0\rangle,  1\rangle\}$	$ +\rangle = \frac{1}{\sqrt{2}}( 0\rangle +  1\rangle)$ $ -\rangle = \frac{1}{\sqrt{2}}( 0\rangle -  1\rangle)$	$ +i\rangle = \frac{1}{\sqrt{2}}( 0\rangle + i 1\rangle)$ $ -i\rangle = \frac{1}{\sqrt{2}}( 0\rangle - i 1\rangle)$
$\{ +\rangle,  -\rangle\}$	$ 0\rangle = \frac{1}{\sqrt{2}}( +\rangle +  -\rangle)$ $ 1\rangle = \frac{1}{\sqrt{2}}( +\rangle -  -\rangle)$	$ +i\rangle = \frac{1}{\sqrt{2}}( +\rangle - i -\rangle)$ $ -i\rangle = \frac{1}{\sqrt{2}}( +\rangle + i -\rangle)$
$\{ +i\rangle,  -i\rangle\}$	$ 0\rangle = \frac{1}{\sqrt{2}}( +i\rangle +  -i\rangle)$ $ 1\rangle = \frac{i}{\sqrt{2}}( -i\rangle -  +i\rangle)$	$ +\rangle = \frac{1}{\sqrt{2}}( -i\rangle + i +i\rangle)$ $ -\rangle = \frac{1}{\sqrt{2}}( +i\rangle - i -i\rangle)$



11. What are the possible outcomes, and the probabilities of measuring those outcomes, when a qubit in one of six possible states  $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |+i\rangle, |-i\rangle$  is measured in each of the following bases? Show or explain your work. You may choose any method to calculate the probabilities.

(A)  $\{|0\rangle, |1\rangle\}$  basis

State of the qubit	Possible outcomes	Probability of measuring each outcome	Is the qubit state parallel or perpendicular to the basis states (on the Bloch sphere)?
$ 0\rangle$			
$ +i\rangle$			

(B)  $\{|+\rangle, |-\rangle\}$  basis

State of the qubit	Possible outcomes	Probability of measuring each outcome	Is the qubit state parallel or perpendicular to the basis states (on the Bloch sphere)?
$ 1\rangle$			
$ -\rangle$			

(C)  $\{|+i\rangle, |-i\rangle\}$  basis

State of the qubit	Possible outcomes	Probability of measuring each outcome	Is the qubit state parallel or perpendicular to the basis states (on the Bloch sphere)?

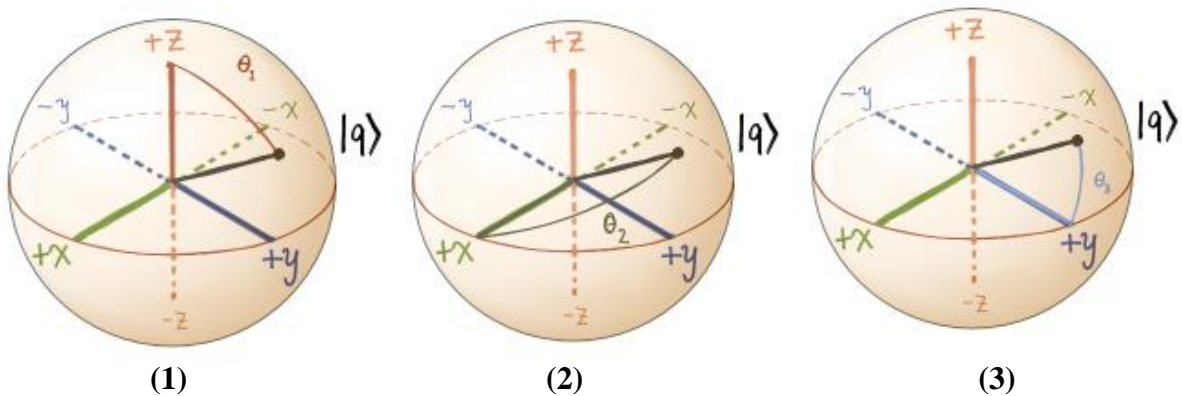
$ +\rangle$			
$ -\rangle$			

*Checkpoint*

- Measurements made in a given state that lies along one of the measurement basis states will yield that basis state with 100% probability.
- Measurements made in a given state that lies perpendicular to the measurement basis states will yield either outcome with 50% probability.

For **questions 12-13**, consider a qubit in the state  $|q\rangle$  illustrated in the following three diagrams, each one highlighting a different measurement basis.

- In diagram (1),  $|q\rangle$  makes an angle  $\theta_1$  with the  $|0\rangle$  state, which lies along the  $+z$  axis.
- In diagram (2),  $|q\rangle$  makes an angle  $\theta_2$  with the  $|+\rangle$  state (the  $+x$  axis).
- In diagram (3),  $|q\rangle$  makes an angle  $\theta_3$  with the  $|+i\rangle$  state (the  $+y$  axis).



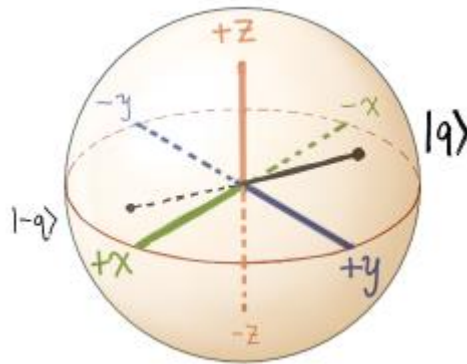
12. Answer parts (A) through (F) in terms of  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ .

- (A) When  $|q\rangle$  is measured in the  $\{|0\rangle, |1\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|0\rangle$ ?
- (B) When  $|q\rangle$  is measured in the  $\{|0\rangle, |1\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|1\rangle$ ?
- (C) When  $|q\rangle$  is measured in the  $\{|+\rangle, |-\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|+\rangle$ ?
- (D) When  $|q\rangle$  is measured in the  $\{|+\rangle, |-\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|-\rangle$ ?

- (E) When  $|q\rangle$  is measured in the  $\{|+i\rangle, |-i\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|+i\rangle$ ?
- (F) When  $|q\rangle$  is measured in the  $\{|+i\rangle, |-i\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|-i\rangle$ ?

If you did the warm-up, compare your answers to [question 8 \[warm-up\]](#) with what you just found for the last few diagrams:  $\theta_1 = \frac{3\pi}{10}$  radians,  $\theta_2 = 1.782$  radians and  $\theta_3 = 0.674$  radians. Do your answers to questions [8](#) and [12](#) match?

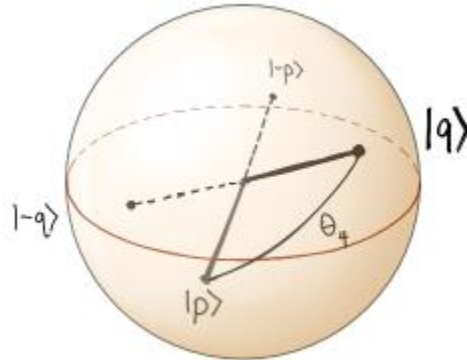
13. Any two states that are found on opposite ends of a diameter of the Bloch sphere can be used as a measurement basis. Let us use the state  $|q\rangle$  and its opposite state,  $|-q\rangle$ , as the measurement basis (illustrated below) for this question.



Answer parts (A) through (F) in terms of  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  as defined in [question 12](#).

- (A) When the state  $|0\rangle$  is measured in the  $\{|q\rangle, |-q\rangle\}$  basis, what are the respective probabilities with which the measurement will yield the state  $|q\rangle$  and the state  $|-q\rangle$ ?
- (B) When  $|1\rangle$  is measured in this basis, what are the respective probabilities with which the measurement will yield the state  $|q\rangle$  and the state  $|-q\rangle$ ?
- (C) When  $|+\rangle$  is measured in this basis, what are the respective probabilities with which the measurement will yield the state  $|q\rangle$  and the state  $|-q\rangle$ ?
- (D) When  $|-\rangle$  is measured in this basis, what are the respective probabilities with which the measurement will yield the state  $|q\rangle$  and the state  $|-q\rangle$ ?
- (E) When  $|+i\rangle$  is measured in this basis, what are the respective probabilities with which the measurement will yield the state  $|q\rangle$  and the state  $|-q\rangle$ ?
- (F) When  $|-i\rangle$  is measured in this basis, what are the respective probabilities with which the measurement will yield the state  $|q\rangle$  and the state  $|-q\rangle$ ?

14. Now also consider the state  $|p\rangle$  and its orthogonal state (the state found opposite to it),  $| -p\rangle$ . The angle between  $|p\rangle$  and  $|q\rangle$  is  $\theta_4$ .



- (A) When  $|q\rangle$  is measured in the  $\{|p\rangle, |-p\rangle\}$  basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?
- (B) When  $|p\rangle$  is measured in the  $\{|q\rangle, |-q\rangle\}$  basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?
- (C) When  $| -p\rangle$  is measured in the  $\{|q\rangle, |-q\rangle\}$  basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?
15. Consider your answers to [questions 12-14](#). What are the cases in which you found the same probabilities? Can you find a pattern?

#### Checkpoint

- When calculating the probability of a given state collapsing into one of the measurement basis states, the only important considerations are the angle between the given state and the states of the measurement basis.
- When  $\theta$  is the angle between a given state and one of the measurement basis states, the probability of the measurement yielding that measurement basis state is  $\cos^2 \frac{\theta}{2}$ , and the probability of it yielding the opposite measurement basis state is  $\sin^2 \frac{\theta}{2} = \cos^2 \frac{\pi - \theta}{2}$ .
- This means that you can always use  $\cos^2 \frac{\theta}{2}$  to determine the probability with which a measurement will yield a target state in the measurement basis, provided that  $\theta$  is the angle between the given state and the target state (not necessarily the angle between the given state and the nearest measurement basis state).
- Any two states that are found on opposite ends of the Bloch sphere can be used as a measurement basis. This means that any direction can be chosen as the z-axis!
- Once a measurement is performed in a qubit state, it is more likely to collapse toward the basis state toward which it “leans closer.”

## Section 2: Geometric intuition

You should be able to:

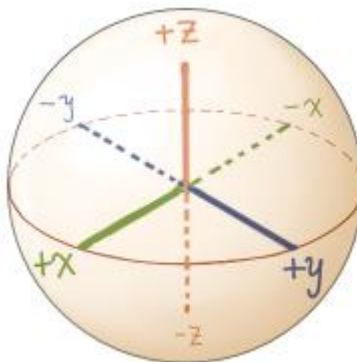
- Identify states for which measurements in a particular basis yield a result with 100% probability or 100% certainty (i.e., eigenstates of the measurement basis)
- Identify that a circle represents the set of all states that a single measurement basis can distinguish from all other states, since the measurement basis cannot determine relative phase in that basis
- Describe that no two distinct measurement bases on the Bloch sphere are compatible; an eigenstate of one basis cannot be an eigenstate of any other basis
- Describe that, in general, a minimum of three measurement bases are needed to completely determine the state of a qubit

### Orientation

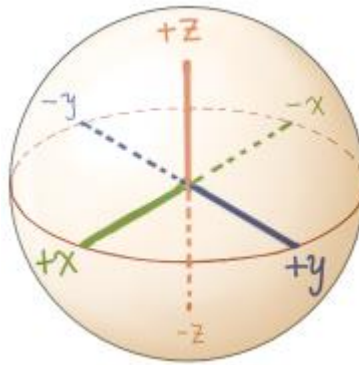
The Bloch sphere is especially useful for thinking about qubit behavior visually. Certain situations are very nice to think about using the Bloch sphere.

If a measurement will yield an outcome with 100% probability, then the measurement is said to be *certain*. This is in contrast to a measurement that might yield either outcome with nonzero probability, which implies some level of *uncertainty* in the measurement.

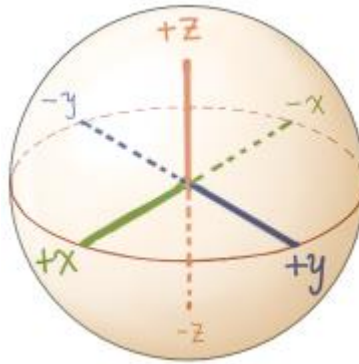
16. In the  $\{|0\rangle, |1\rangle\}$  measurement basis, what are all the states for which a measurement outcome is certain (i.e., the outcome will be measured with 100% probability)? Identify them on the Bloch sphere below.



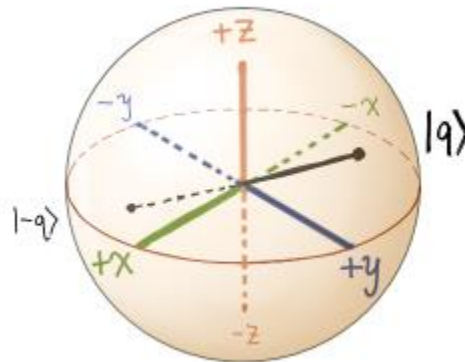
17. In the  $\{|0\rangle, |1\rangle\}$  basis, what is the set of all states that will yield each of the basis states with 50% probability? (Hint:  $|+\rangle$ ,  $|-\rangle$ ,  $|+i\rangle$ , and  $| -i\rangle$  are not the only such states.) Identify the states on the Bloch sphere below.



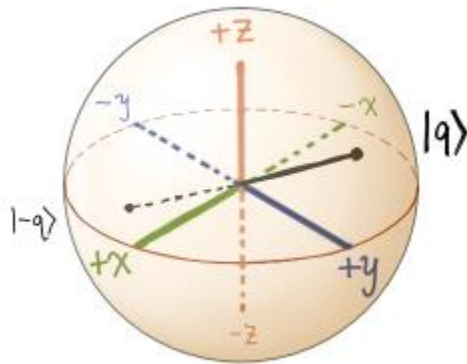
18. In the  $\{|0\rangle, |1\rangle\}$  basis, what is the set of all states that will yield the  $|0\rangle$  state with 70% probability and the  $|1\rangle$  state with 30% probability? Identify the states on the Bloch sphere below. (Make a rough approximation.)



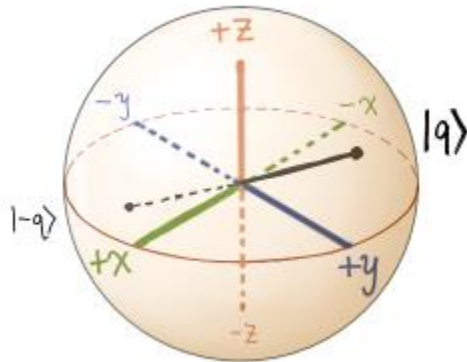
19. In the  $\{|q\rangle, |-q\rangle\}$  basis, what are all the states for which the measurement outcome is certain? Identify the states on the Bloch sphere below.



20. In the  $\{|q\rangle, |-q\rangle\}$  basis, what is the set of all states that will yield each of the basis states with 50% probability? Identify the states on the Bloch sphere below.

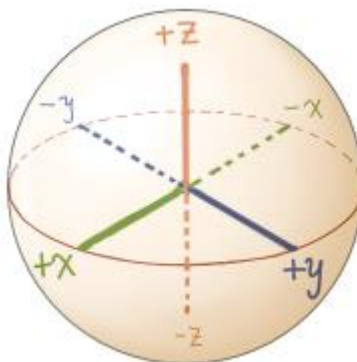


21. In the  $\{|q\rangle, |-q\rangle\}$  basis, what is the set of all states that will yield the  $|q\rangle$  state with 90% probability and the  $|-q\rangle$  state with 10% probability? Identify the states on the Bloch sphere below. (Make a rough approximation.)

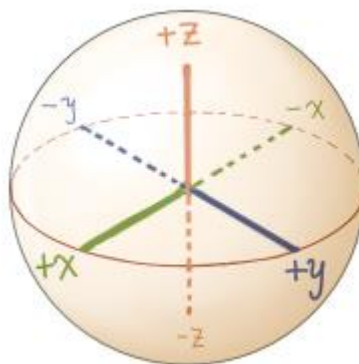


22. When a measurement is made in either the  $\{|0\rangle, |1\rangle\}$  or  $\{|+\rangle, |-\rangle\}$  basis, does there exist some state for which both measurements are certain (i.e. yield some outcome with 100% probability)? We will try to reach an answer in the following parts to this question.

(A) As in [question 16](#), draw on the Bloch sphere below all of the states for which a measurement outcome is certain in the  $\{|0\rangle, |1\rangle\}$  basis.



(B) Draw on the Bloch sphere below all of the states for which a measurement outcome is certain in the  $\{|+\rangle, |-\rangle\}$  basis.

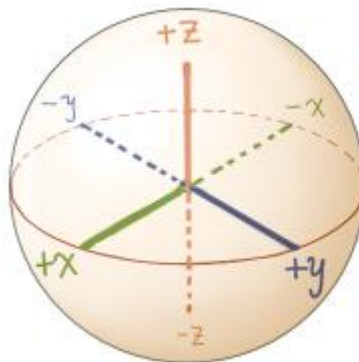


(C) Are there any states that are members of both sets?

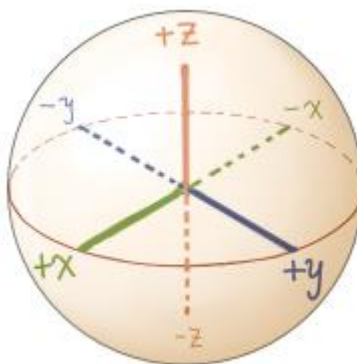
(D) Generalize this to other bases. For *any* two different bases, is it possible to find a state that yields a 100% certain outcome when a measurement is made in each basis (on identically-prepared qubits—not in succession)?

23. Instead of 100% certainty, is it possible for a measurement to yield either outcome with 50% probability in multiple bases at once? We will find out in the following parts.

(A) On the Bloch sphere below, first draw the set of states that, when measured in the  $\{|0\rangle, |1\rangle\}$  basis, yields each outcome with 50% probability. Then draw the set of states that yields each outcome with 50% probability in the  $\{|+\rangle, |-\rangle\}$  basis. Indicate clearly any points of overlap.



(B) On the Bloch sphere below, first draw the set of states that, when measured in the  $\{|+\rangle, |-\rangle\}$  basis, yields each outcome with 50% probability. Then draw the set of states that yields each outcome with 50% probability in the  $\{|+i\rangle, |-i\rangle\}$  basis. Indicate clearly any points of overlap.



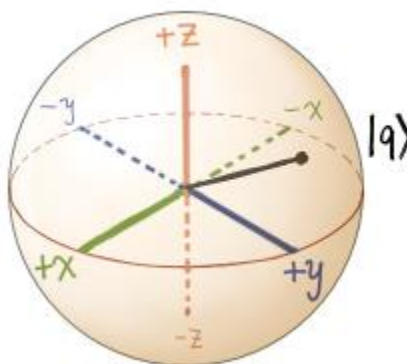
(C) Does there exist some state on the Bloch sphere that, when measured in all three of the  $\{|0\rangle, |1\rangle\}$ ,  $\{|+\rangle, |-\rangle\}$ , and  $\{|+i\rangle, |-i\rangle\}$  bases (which are all independent), yields each outcome with 50% probability in each basis?

(To answer this question, look at your two diagrams for the previous two problems. Are there any points where all three sets of states should overlap? I.e., are there points of overlap that are shared between the two diagrams?)

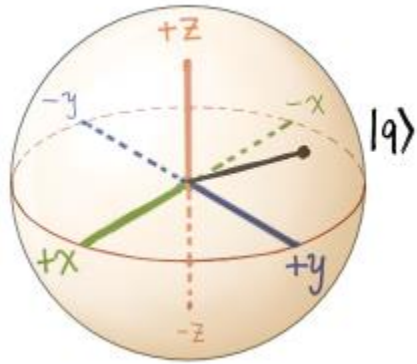
*Note: This tutorial only discusses pure states, which are represented by points on the surface of the Bloch sphere. The answer may be different if one starts with a mixed state—but we will not deal with mixed states.*

24. Measurements can be used to determine where a state lies on the Bloch sphere. Let us illustrate the point by working backwards. Assume that we have a well-defined state, such as  $|q\rangle$  illustrated below on the Bloch sphere.

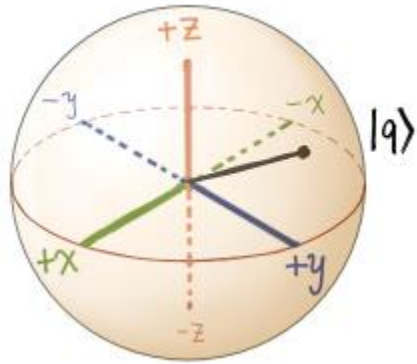
(A) Draw the set of all states that are indistinguishable from  $|q\rangle$  when measured in the  $\{|0\rangle, |1\rangle\}$  basis, in the sense that they would yield the  $|0\rangle$  state with the same probability that  $|q\rangle$  would yield the  $|0\rangle$  state.



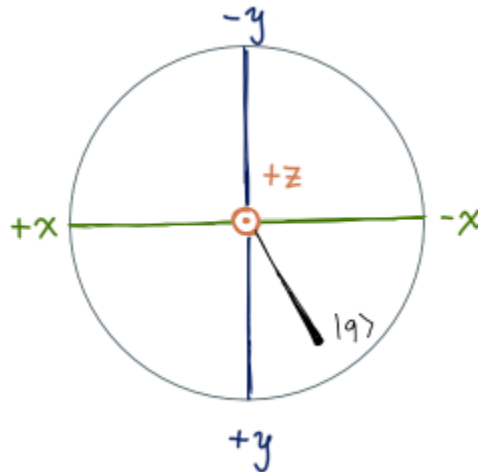
(B) Do the same as above with the  $\{|+\rangle, |-\rangle\}$  basis: Draw the set of all states that are indistinguishable from  $|q\rangle$  when measured in the  $\{|+\rangle, |-\rangle\}$  basis.



(C) Do the same as above with the  $\{|+i\rangle, |-i\rangle\}$  basis: Draw the set of all states that are indistinguishable from  $|q\rangle$  when measured in the  $\{|+i\rangle, |-i\rangle\}$  basis.



25. The three-dimensional Bloch sphere is difficult to visualize, draw on, and interpret on a two-dimensional surface. The projection below looks down on the Bloch sphere from the north pole (+z axis).



On this projection:

(A) Draw the set of all states that that serves as your answer to question 24(A).

*Hint:* In this view, your answer will be a circle.

(B) Draw your answer to question 24(B).

*Hint:* This is a circle viewed from the side, so it will look like a straight line. Since we deal with states only on the surface of the Bloch sphere, this line will reach the circumference at both ends.

Is this enough to pin down the location of the state  $|q\rangle$ ? (*Hint:* At how many points do these two circles intersect?)

(C) Draw your answer to part 24(C), which will look similar to part (B). Is this enough to pin down the location of the state  $|q\rangle$ ?

(D) How many measurement bases did it take to locate this state on the Bloch sphere? In general, how many different (orthonormal) measurement bases are necessary to fully determine any state?

(E) Are there any special cases in which fewer or more than this number of measurement bases is necessary? (You may want to return to thinking about the 3-D Bloch sphere instead of a 2-D cross-section.)

*Case 1:* Can two circles intersect at fewer than two points? What would this case look like?

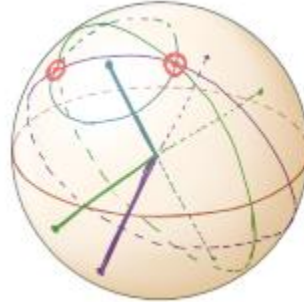
*Case 2:* Suppose the circle of the first measurement has a radius of zero. What would this case look like?

*Case 3:* Suppose two circles are drawn that intersect at two points. Can you draw an additional circle that does not narrow the number of points down from two to one? How many such additional circles can be drawn, if any?

(F) Is it possible for any of these circles to not intersect at all? Why or why not?

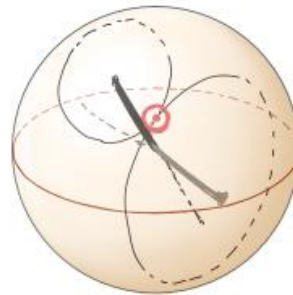
Comment: The necessity of making three independent measurements is analogous to how the GPS (global positioning system) works. A minimum of three satellites are consulted, and each one measures a distance from itself to the transceiver. Since each satellite cannot reliably determine direction, the transceiver can be anywhere on a spherical surface centered on the satellite, with the distance as the radius. The first satellite's surface intersects the Earth with a circle; the second narrows it down to two points; and the third picks out one of the two points.

The “independence” of these measurements refers to the linear independence of the measurement bases that are chosen. If the bases all lie in the same plane, then they do not cover all of the degrees of freedom, and no number of such bases will be sufficient to determine the state.



*It is possible for three or more measurement bases (blue, green, and purple) to result in circles that still overlap at two points. In this case, the three measurement bases are coplanar, and thus not linearly independent.*

On the other hand, if the measurement bases are chosen such that the state of interest lies directly upon one of the measurement basis states, or that one of the circles lies tangent to another, then it is possible that fewer than 3 measurement bases are necessary to determine the state. These cases are highly unlikely if the state is not known beforehand.



*Circles lying tangent to each other can narrow the state down to a single point on the Bloch sphere after measurement in only two bases!*

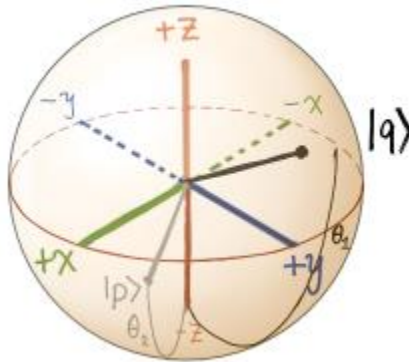
#### Checkpoint

- Three separate measurement bases are necessary to determine where a state lies on the Bloch sphere. In most cases, two circles will intersect at two points; a third circle, corresponding to the third measurement, selects one of the two points.
- A measurement made in a particular basis cannot detect relative phase with respect to that basis (i.e., the measurement can only determine a value of  $\theta$ , but not  $\phi$ ), which is why a single measurement basis narrows down the state only to a circle of constant latitude.

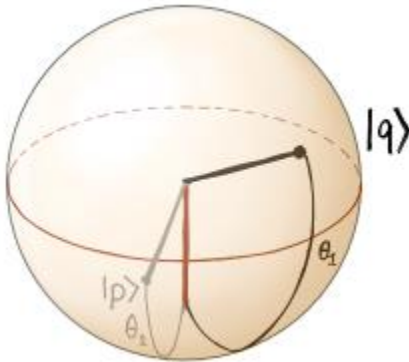
## Addendum to Section 2: Successive measurements

You should be able to describe how a series of measurements made in immediate succession may not yield outcomes with the same probabilities as a measurement made in the initial state in the final chosen basis

26. Consider a qubit in the state  $|1\rangle$ , as illustrated on the Bloch sphere below.  $|1\rangle$  makes an angle  $\theta_1$  with  $|q\rangle$  and an angle  $\theta_2$  with  $|p\rangle$ .

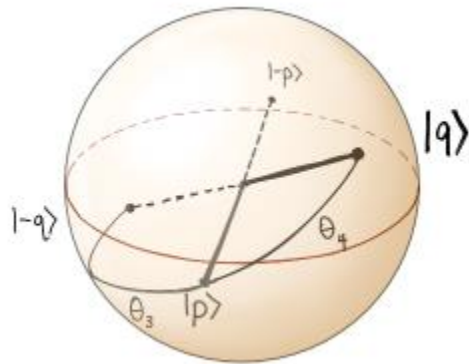


- (A) When  $|1\rangle$  is measured in the  $\{|q\rangle, |-q\rangle\}$  basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?



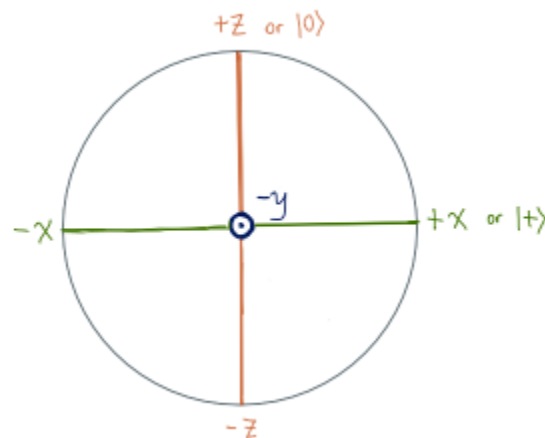
- (B) When  $|1\rangle$  is measured in the  $\{|p\rangle, |-p\rangle\}$  basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?
- (C) Imagine that the  $|1\rangle$  state is measured first in the  $\{|p\rangle, |-p\rangle\}$  basis, yielding the outcome  $|p\rangle$ . We then make a measurement in immediate succession in the  $\{|q\rangle, |-q\rangle\}$  basis. For this second measurement, what are the outcomes and respective probabilities of measuring those outcomes?

For reference, the angles between the  $\{|q\rangle, |-q\rangle\}$  basis states and the  $\{|p\rangle, |-p\rangle\}$  basis states are defined below.



(D) Imagine that the  $|0\rangle$  state is measured first in the  $\{|p\rangle, |-p\rangle\}$  basis, yielding the outcome  $|p\rangle$ . We then make a measurement in immediate succession in the  $\{|q\rangle, |-q\rangle\}$  basis. For this second measurement, what are the outcomes and respective probabilities of measuring those outcomes?

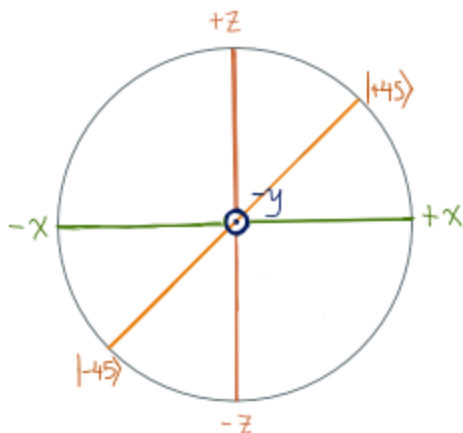
27. Consider a qubit in the state  $|0\rangle$ , as seen on the cross-section (viewed down the  $y$ -axis) of the Bloch sphere below.



(A) When  $|0\rangle$  is measured directly in the  $\{|+\rangle, |-\rangle\}$  basis, what are the outcomes of the measurement, and what are the respective probabilities of measuring those outcomes?

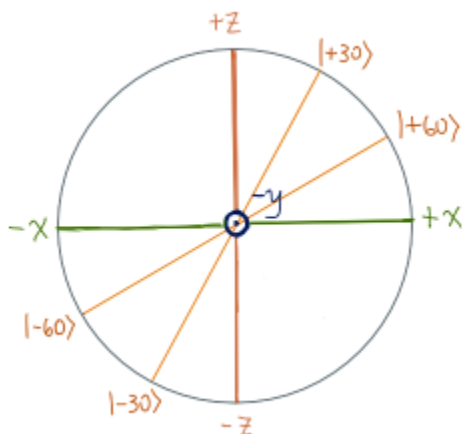
(B) Illustrated below is a separate basis, called here the  $\{|+45\rangle, |-45\rangle\}$  basis, whose diametrically opposite basis states point at  $45^\circ$  from  $|0\rangle$  and  $|1\rangle$  respectively in the  $x$ - $z$  plane.

Consider a case where  $|0\rangle$  is measured first in this basis before being measured in the  $\{|+\rangle, |-\rangle\}$  basis.



- i. If the first measurement yields the  $|+45\rangle$  state, what are the outcomes of the second measurement made in the  $\{|+\rangle, |-\rangle\}$  basis, and what are the respective probabilities of measuring those outcomes?
- ii. If the first measurement yields the  $|-45\rangle$  state, what are the outcomes of the second measurement made in the  $\{|+\rangle, |-\rangle\}$  basis, and what are the respective probabilities of measuring those outcomes?
- iii. Let us return to the original problem of using measurements to sequentially collapse to states  $|0\rangle \rightarrow |?\rangle \rightarrow |+\rangle$ , where  $|?\rangle$  represents the result of a measurement made in the  $\{|+45\rangle, |-45\rangle\}$  basis. How many possible ways are there to achieve the final state of  $|+\rangle$ , and are their respective overall probabilities of occurring similar or different?

(C) We would still like to end with a final state of  $|+\rangle$ , but this time after three successive measurements instead of two. This time the bases, shown below, are  $30^\circ$  apart from one another instead of  $45^\circ$ :



Calculate the probability of the state  $|0\rangle$  collapsing into the state  $|+\rangle$  after three measurements, using the sequence  $|0\rangle \rightarrow |+30\rangle \rightarrow |+60\rangle \rightarrow |+\rangle$ . (If we also allow the qubit to collapse into the intermediate states  $|-30\rangle$  and  $|-60\rangle$ , does this increase or decrease the probability of collapsing into  $|+\rangle$  at the end of three measurements?)

(D) Make a qualitative argument for how to increase the probability of ultimately collapsing the state  $|0\rangle$  into the state  $|+\rangle$ , through measurements alone.

*Checkpoint*

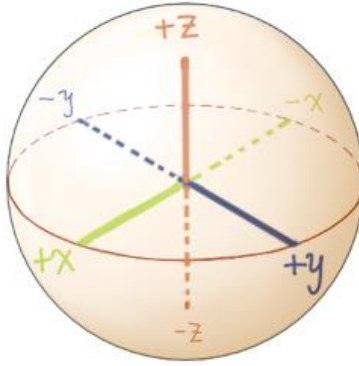
- Making multiple successive measurements on a qubit in different bases will not give the same probability of measuring the desired outcome as will making a single measurement in the given state in the final basis.
- When successive measurements are made, the outcomes will depend only on the state of the qubit obtained at the end of the immediately preceding measurement. The overall outcome depends on the entire process.
- The formula for finding the probability of collapsing to the state  $|+\rangle$  from the given state  $|0\rangle$  after  $n$  measurements in evenly-spaced bases is  $\left(\cos^2 \frac{\pi}{4n}\right)^n$ . Plotting this as a function of  $n$  in your favorite piece of software (e.g., Mathematica, Desmos, etc.) reveals that, for  $n \geq 1$ , this is a monotonically increasing function that asymptotically approaches a probability of 1. This means that increasing the number of intermediate measurements improves the chances of collapsing into the desired state.

This is a similar idea to using a series of linear polarizing filters to rotate the polarization of light.

## Appendix E.2 Pre-test and post-test

Notes:

- In the Bloch sphere diagrams of this section, the  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$  unit vectors point toward the viewer, while the  $-\hat{x}$ ,  $-\hat{y}$  and  $-\hat{z}$  unit vectors point away from the viewer.
- Solid lines indicate states on the front hemisphere (which point toward the viewer), while dotted lines indicate states on the back hemisphere.
- When a measurement basis is indicated, the larger ket label indicates the closer state (which points toward the viewer), while the smaller ket label indicates the state further away.



- In these questions, states are said to be “equivalent” if they yield identical measurement outcomes with identical probabilities for all observables (i.e., in all measurement bases).
- A state  $|q\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$  can be found on the Bloch sphere at the position of the Cartesian vector  $\mathbf{r} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$ , where  $0 \leq \theta \leq \pi$  and  $0 \leq \phi < 2\pi$ .
  - The polar angle  $\theta$  begins from the positive  $z$ -axis and sweeps toward the equator, and the azimuthal angle  $\phi$  begins from the positive  $x$ -axis and sweeps counterclockwise about the  $z$ -axis.

### Warm-up

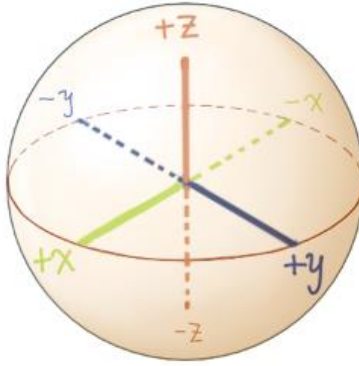
Learning objectives: Students should be able to identify that multiplying by an overall phase does not change the state, but modifying a relative phase between the basis states does

1. Consider whether the following pairs of states are equivalent, i.e., states that yield identical measurement outcomes with identical probabilities in all bases:
  - (A)  $|q\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$  and  $|p\rangle = \frac{i\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$ . Are  $|q\rangle$  and  $|p\rangle$  equivalent states?
  - (B)  $|b\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$  and  $|d\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$ . Are  $|b\rangle$  and  $|d\rangle$  equivalent states?
2. True or false: Any normalized state written as  $a|0\rangle + b|1\rangle$  corresponds to a unique point on the Bloch sphere.

### Measurements

Learning objectives: Students should be able to indicate a state on the Bloch sphere given  $\theta$  and  $\phi$ ; identify the outcomes of measurement when presented with a generic state  $|q\rangle$  on the Bloch sphere with  $\{|0\rangle, |1\rangle\}$  being the measurement basis; describe the measurement outcomes and calculate the probabilities of those outcomes for the given generic state  $|q\rangle$  when  $\{|0\rangle, |1\rangle\}, \{|+\rangle, |-\rangle\}, \{|+i\rangle, |-i\rangle\}$  are each used as measurement bases (by changing basis)

3. Consider the state  $|q\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}e^{i\frac{2\pi}{3}}|1\rangle$ .
  - (A) On the Bloch sphere below, indicate the state  $|q\rangle$ . Label  $\theta$  and  $\phi$ .

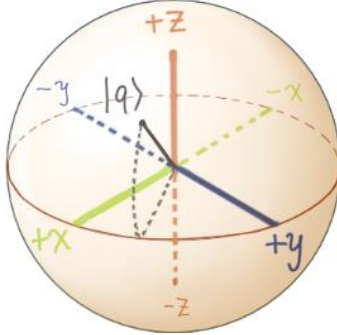


(B) The state  $|q\rangle = \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}e^{i\frac{2\pi}{3}}|1\rangle$  is measured in the  $\{|0\rangle, |1\rangle\}$  basis. What are the outcomes? What is the probability of yielding each outcome? (There is no need to find numerical values.)

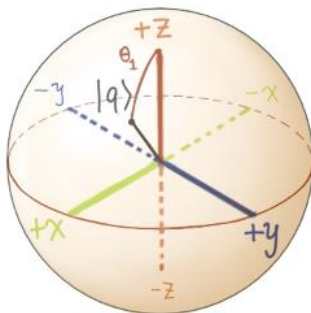
(C) The state  $|q\rangle$  is measured in the  $\{|+\rangle, |-\rangle\}$  basis. Which outcome is more probable?

(D) The state  $|q\rangle$  is measured in the  $\{|+i\rangle, |-i\rangle\}$  basis. Which outcome is more probable?

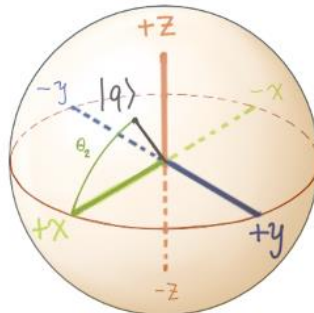
4. Consider a qubit in the state  $|q\rangle = a|0\rangle + b|1\rangle$  shown below.



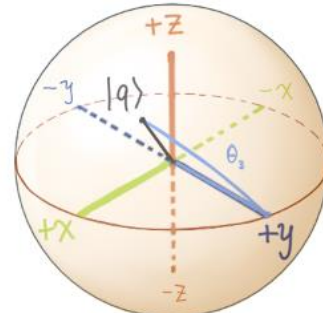
$|q\rangle$  is illustrated in the following three diagrams, each one highlighting a different measurement basis. The (nonstandard) notations  $\theta_1, \theta_2,$  and  $\theta_3$  will be used to refer to the angles made with respect to one of the states in this measurement bases.



(I)



(II)



(III)

(A) In diagram (I),  $|q\rangle$  makes an angle  $\theta_1$  with the  $|0\rangle$  state.

(B) In diagram (II),  $|q\rangle$  makes an angle  $\theta_2$  with the  $|+\rangle$  state.

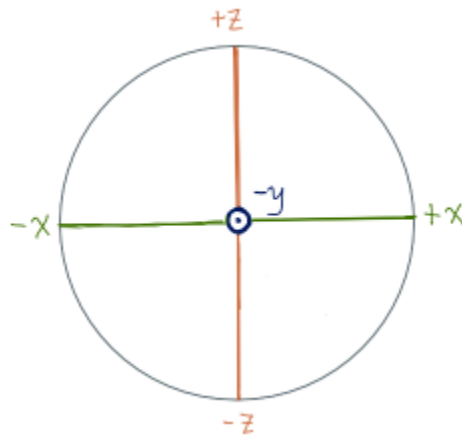
(C) In diagram (III),  $|q\rangle$  makes an angle  $\theta_3$  with the  $|+i\rangle$  state.

Express your answers to (A) and (B) in terms of  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ .

(A) When  $|q\rangle$  is measured in the  $\{|+\rangle, |-\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|+\rangle$ ?

(B) When  $|q\rangle$  is measured in the  $\{|+i\rangle, |-i\rangle\}$  basis, what is the probability that  $|q\rangle$  will collapse into the state  $|-i\rangle$ ? (Note: **NOT**  $|+i\rangle$ !)

5. Below is shown a cross-section of the Bloch sphere, with the axes indicated.



(A) On the cross-section of the Bloch sphere, indicate a state of your choosing that, when measured in the  $\{|0\rangle, |1\rangle\}$  basis, has a higher chance of yielding  $|0\rangle$  than  $|1\rangle$ .

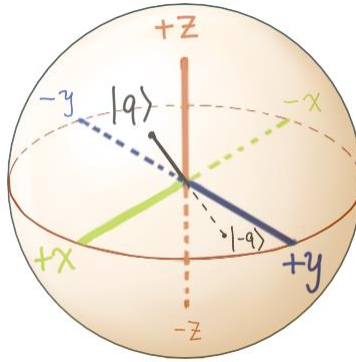
(B) On the cross-section of the Bloch sphere, indicate a state of your choosing that, when measured in the  $\{|0\rangle, |1\rangle\}$  basis, has a higher chance of yielding  $|1\rangle$  than  $|0\rangle$ .

(C) On the cross-section of the Bloch sphere, indicate a state of your choosing that, when measured in the  $\{|+\rangle, |-\rangle\}$  basis, has a higher chance of yielding  $|+\rangle$  than  $|-\rangle$ .

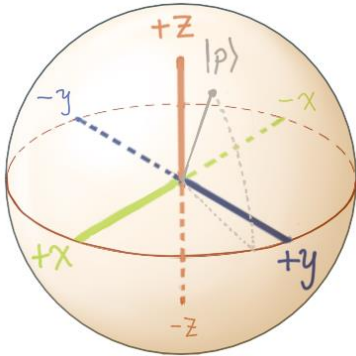
### Geometric intuition

Learning objectives: Students should be able to identify that every basis has only two orthonormal states that can yield results with 100% certainty; describe that no two distinct measurement bases on the Bloch sphere are compatible (an eigenstate of one basis cannot be an eigenstate of any other basis); identify that a circle represents the set of all states that a single measurement basis can distinguish from all other states, since the measurement basis cannot determine relative phase in that basis

6. On the Bloch sphere below, the  $\{|q\rangle, |-q\rangle\}$  basis is illustrated. Indicate (or otherwise specify) all the states in which a measurement in this basis yields a result with 100% certainty.

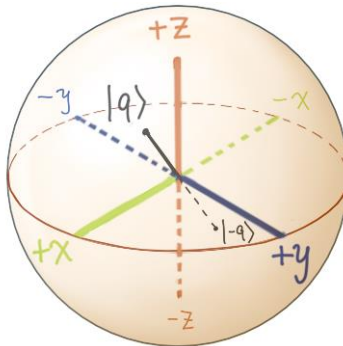


7. Consider the state  $|p\rangle$  on the Bloch sphere below.



You are in state  $|p\rangle$  and want to make a measurement such that you get a certain outcome with 100% probability. How many measurement bases can you choose to accomplish this?

- a. 0
  - b. 1
  - c. 2
  - d. Infinitely many
8. On the Bloch sphere below, indicate the set of all states that, when measured in the illustrated  $\{|q\rangle, |-q\rangle\}$  basis, will yield  $|q\rangle$  with (approximately) 90% probability and  $|-q\rangle$  with 10% probability.



## Appendix F Materials for Chapter 7

### Appendix F.1 Tutorial on the basics of quantum computing

#### **Basics of classical and quantum computers**

*\*Notes:*

*In this tutorial, any use of the word “bit” will refer to a classical bit, and quantum bits are unambiguously referred to in all cases as “qubits.”*

*Typical notation for quantum information will be used; their counterparts in typical quantum physics notation are listed below:*

$$\begin{aligned} \{|+z\rangle, |-z\rangle\} &\rightarrow \{|0\rangle, |1\rangle\} \\ \{|+x\rangle, |-x\rangle\} &\rightarrow \{|+\rangle, |-\rangle\} \\ \{|+y\rangle, |-y\rangle\} &\rightarrow \{|+i\rangle, |-i\rangle\} \end{aligned}$$

*Key*

- For Roman numerals in parentheses (I), (II), etc., choose all the correct answers.
- For lowercase letters followed by periods a., b., c., etc., choose a single answer.
- Uppercase letters in parentheses (A), (B), etc. indicate multiple parts of a free-response question. Answer all of them.

#### ***Introduction***

Classical computers and quantum computers: What’s the difference?

Classical computers process classical information, and quantum computers process quantum information. Classical information is stored in bits, and quantum information is stored in qubits. Quantum information is more difficult to manipulate than classical information, but people have found clever ways to use it to solve problems.

Classical computers have limits: Some problems demand enormous computational power as the system size increases, and it would take many times longer than the age of the universe to solve such a problem on a classical computer. For some (but only *some*) of these problems, quantum computers can bring the computation time down significantly.

#### **Section 1: Comparisons between classical and quantum computers**

##### ***Introduction***

Quantum computing is an exciting topic, but also one that involves some new concepts. To better understand quantum computers, let us first compare them to classical computers.

##### **Mini-section 1: Classical bits**

*Learning objectives:*

*You should be able to:*

- *Define a bit*
- *Identify the states of multi-bit systems*
- *Write down the number of possible states for an N-bit system*

*Identify what physical systems can be used to represent bits*

### *Orientation*

A computer processes **information**. To do this, it has an input device (e.g., a keyboard), a processor, memory, and an output device (e.g., a computer screen).

Classical computers process classical information encoded in **bits**. But what exactly are bits? Let's first look at some of their properties.

### *Questions*

1. Choose all of the following statements that are true:

- (I) A bit is used to encode the smallest unit of information in a classical computer
- (II) A single bit can hold only one of two values, which can be labeled as 0 and 1.
- (III) Any information can be represented by combining bits into larger aggregates.

28. Choose all of the following that can be used as a bit:

- (I) Any measuring device
- (II) Powers of 10
- (III) Something that can be in one of two distinct states

29. Choose all of the following that can be used as a bit:

- (I) "Left" and "right" orientations of the magnetic moment of a magnetic domain
- (II) Voltage across a capacitor, as measured to be above and below a cutoff value
- (III) Heads and tails of a coin

The *heads* side and *tails* side of a coin can be used to represent a bit. For instance, we could label heads as "0" and tails as "1," or vice versa. A coin that is neither heads nor tails, such as one standing on its edge, cannot be used as a bit. Other properties of the coin that are not binary, such as its color or metal content, cannot be used to represent a bit.

Other objects can also be used to represent bits. For example, in a classical computer, a capacitor is charged to a voltage either *above* or *below* some predetermined threshold. One of these states is considered the "0" state, and the other the "1" state.

30. One bit can be in one of two distinct states. How many distinct states does a 2-bit system have?

- a. 2
- b. 3
- c. 4

Write out all the possible states of a 2-bit system. For example, the state in which both bits take the value 0 can be written as “00”.

31. How many distinct states does a 3-bit system have?

- a. 6
- b. 8
- c. 9

Write out all the possible states, e.g., “000,” ...

32. How many distinct states does an  $N$ -bit system have?

- a.  $2N$
- b.  $N^2$
- c.  $2^N$

Write out a few such states, e.g., “000 ... 0,” “000 ... 1,” ...

#### Checkpoint

- A **bit**, which is a contraction of the words *binary digit*, can take one of two values, 0 or 1. Computation involving bits uses a base 2 numbering system.
  - In everyday life, we count from 0 to 9 before increasing the digit immediately to the left by 1, then restarting the ending digit at zero. In base 2, we count from 0 to only 1 before increasing the next leftward digit.
- A bit can be encoded in any sort of system that exists in one of two mutually exclusive states. Whatever physical entity these bits are encoded in, a computer with  $N$  of these bits can represent any base 2 number with  $N$  digits or fewer. Even a modest  $N$  can create many combinations (possible states). A 64-bit computer works with  $2^{64} \sim 10^{19}$ : ten quintillion possible states!
- Depending on the desired calculation, a computer can run a set of logic operations on the input to get its output (or results).

## Mini-section 2: Quantum bits (qubits)

You should be able to:

- Define a qubit
- Describe what physical systems can be used to represent qubits.
- Identify linearly independent states of qubit systems.
- Write down the number of linearly independent states given by  $N$  qubits
- Identify that probabilities of measurement outcomes on a qubit must add up to 1.

### Orientation

Classical (regular) computers may encode and process information in bits, but quantum computers encode and process quantum information in *quantum bits*, or **qubits**.

The **state** of a single qubit can be represented as  $|q\rangle = a|0\rangle + b|1\rangle$ , with  $|0\rangle$  and  $|1\rangle$  being the basis states and  $a$  and  $b$  being complex numbers with  $|a|^2 + |b|^2 = 1$ . Because it always takes two basis states to describe a qubit, a qubit is an example of a two-state system. The state of a qubit can be taken as a description of the qubit's condition, but it is often useful to think of the state as a vector in a two-dimensional Hilbert space.

### Answer this:

*How are quantum computers similar to classical computers? How are they different? Describe in your own words, using no more than 2-3 sentences.*

An electron spin state can be used to represent a qubit, with the two mutually exclusive possibilities being the *spin-up* (which could be analogous to heads) and *spin-down* (tails) states. An electron has other interesting properties, but any property that cannot be mapped onto two linearly independent (or mutually exclusive) states cannot be used to represent a qubit.

### Questions

33. Choosing basis states for a vector space is analogous to choosing...

- (I) A coordinate system
- (II) Operators
- (III) Eigenvalues

34. Choose all of the following physical systems that can be used as a qubit.

- (I) The eigenstates of  $\hat{S}_z$  for a spin  $\frac{1}{2}$  particle
- (II) Two orthogonal polarization states of a photon
- (III) A particle that can only be in the ground state and the first excited state of a 1-dimensional infinite square well

35. Consider the following conversation between two students.

**Student 1:** There are only two possible electron spin states. The electron can either be spin-up or spin-down.

**Student 2:** No! Those are two linearly independent states. But a qubit formed with electron spin states can in general be in a superposition of those two states, e.g.,  $\frac{1}{4}|0\rangle + \frac{\sqrt{15}}{4}|1\rangle$ , with  $|0\rangle$  and  $|1\rangle$  being the spin-up and spin-down states.

Which student(s), if any, do you agree with? Explain.

36. How many orthogonal states are required to form a complete basis for a system that consists of 2 qubits?

- a. 2
- b. 3
- c. 4

Write out all the possible states using the notation of kets, e.g., the state in which both qubits take the value 0 can be written as  $|00\rangle$ , ...

37. How many orthogonal states are required to form a complete basis for a 3-qubit system?

- a. 3
- b. 6
- c. 8

Write out all the possible basis states using the notation of kets, e.g.,  $|000\rangle$ , ...

38. How many orthogonal states are required to form a complete basis for an  $N$ -qubit system?

- a.  $N$
- b.  $2N$
- c.  $2^N$

Write out a few such basis states using the notation of kets, e.g.,  $|000 \dots 0\rangle$ , ... Compare your qubit basis states to your bit states. How are they similar or different?

*A word on notation:*

The symbol  $|00\rangle$  is a contraction, representing a state for the two-qubit system in which both the first and second qubits are in the  $|0\rangle$  states:  $|0\rangle_1 \otimes |0\rangle_2$ . Likewise,  $|0\rangle_1 \otimes |1\rangle_2$  represents the state of a two-qubit system in which the first qubit is in the  $|0\rangle$  state and the second qubit is in the  $|1\rangle$  state:

$$\begin{aligned}|0\rangle_1 \otimes |0\rangle_2 &= |0\rangle \otimes |0\rangle = |00\rangle \\ |0\rangle_1 \otimes |1\rangle_2 &= |0\rangle \otimes |1\rangle = |01\rangle \\ |1\rangle_1 \otimes |0\rangle_2 &= |1\rangle \otimes |0\rangle = |10\rangle \\ |1\rangle_1 \otimes |1\rangle_2 &= |1\rangle \otimes |1\rangle = |11\rangle\end{aligned}$$

Similarly, for a three-qubit system, the state in which the first, second, and third qubits are in the state  $|0\rangle$  is represented by  $|0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 = |0\rangle \otimes |0\rangle \otimes |0\rangle = |000\rangle$ , and so on.

39. Imagine the outcome of a binary event, i.e., the outcome can be one of only two possibilities. The probability that outcome A occurs is  $P(A)$ .

Which of the following *must* be true of the probability of occurrence of outcome B,  $P(B)$ ?

- (I)  $P(B) > P(A)$
- (II)  $P(B) = P(A)$
- (III)  $P(B) = 1 - P(A)$

40. Consider the following conversation between several students.

**Student 1:** An  $N$ -bit classical computer has access to  $N$  available states during its calculation, but an  $N$ -qubit quantum computer has  $2^N$  linearly independent, mutually exclusive basis states.

**Student 2:** That's not right. Similarly to the quantum computer's basis states, the classical computer also has  $2^N$  states, remember? Each of the  $N$  bits could be 0 or 1, so the total number of possibilities for an  $N$ -bit classical computer state is also  $2^N$ .

**Student 1:** I thought that the quantum computer can be in more states than the classical computer! Now you're saying that both an  $N$ -bit classical computer and an  $N$ -qubit quantum computer can only be in one of  $2^N$  possible states?

**Student 2:** No, that's not what I'm saying. If you think of a single bit, so  $N = 1$ , that bit can be in one of 2 states, but a single qubit can be in a continuum of possible states, each written as a superposition of 2 linearly independent basis states— $a|0\rangle + b|1\rangle$ . There are infinitely many possibilities because each of  $a$  and  $b$  can be any complex number with modulus between zero and one, so long as  $|a|^2 + |b|^2 = 1$ . So an  $N$ -qubit quantum computer can also be in one among a continuum of states:  $a_0|000 \dots 0\rangle + a_1|000 \dots 1\rangle + \dots$ .

**Student 3:** So, then, an  $N$ -bit classical computer can be in one of  $2^N$  states, and an  $N$ -qubit quantum computer can be in one of infinitely many states, which can all be written as a superposition of the  $2^N$  linearly independent basis states  $|000 \dots 0\rangle, |000 \dots 1\rangle$ , etc.

Which student(s), if any, do you agree with? Explain.

<i>Checkpoints</i>	
<ul style="list-style-type: none"> <li>A (classical) bit can take two values, 0 and 1.</li> </ul> <p><i>Note: When the word “bit” is used in this tutorial, assume that it refers to a classical bit.</i></p>	<ul style="list-style-type: none"> <li>A qubit can be represented as <math> q\rangle = a 0\rangle + b 1\rangle</math>, with <math> 0\rangle</math> and <math> 1\rangle</math> being the orthonormal basis states that form the standard (computational) basis,* and <math> a ^2 +  b ^2 = 1</math>.</li> </ul>
<ul style="list-style-type: none"> <li>A bit is what a computer uses to represent classical information. Bits store information in the form of a binary (base 2) number.</li> </ul>	<ul style="list-style-type: none"> <li>A qubit is what a quantum computer uses to represent quantum information, and when measured in the standard basis, it takes one of two values (<math> 0\rangle</math> or <math> 1\rangle</math>).</li> </ul>
<ul style="list-style-type: none"> <li>At a given time, an <math>N</math>-bit system can be in one of <math>2^N</math> distinct states, with each of these <math>2^N</math> states being a specific binary number.</li> </ul>	<ul style="list-style-type: none"> <li>At a given time, an <math>N</math>-qubit system can be in one of infinitely many states, but each state can be written as a superposition of <math>2^N</math> linearly independent (or mutually exclusive) basis states.</li> </ul>
<ul style="list-style-type: none"> <li>Choosing a basis is analogous to choosing a coordinate system, or choosing a set of unit vectors spanning the space.</li> <li>The number of possible distinct states, <math>2^N</math>, for a system of <math>N</math> classical bits is the same as the number of linearly independent basis states (or measurement outcomes) for a system of <math>N</math> qubits (also <math>2^N</math>). In either case, one obtains <math>N</math> bits of information from each state.</li> </ul>	
<ul style="list-style-type: none"> <li>A classical computer presents a string of <math>N</math> bits as its output to a computation.</li> </ul>	<ul style="list-style-type: none"> <li>For a system in a quantum computer, one can use a superposition of <math>N</math> qubits to do computations. However, the output of a quantum computer is also a string of <math>N</math> classical bits.</li> </ul> <p>(In other words, we can only get <math>N</math> classical bits of information from an <math>N</math>-qubit quantum computer at the end of the computation.)</p>

\*Note: The **standard basis** or **computational basis** are other names for the  $\{|0\rangle, |1\rangle\}$  basis for a single qubit.

## Section 2: Measuring qubits

### **Answer this:**

*In 2-3 sentences, describe some of the similarities and differences between classical and quantum computers.*

### **Introduction**

Quantum measurements are conducted in a particular basis. A **measurement basis** for a qubit consists of two orthonormal basis states. Since qubit states are vectors in a vector space, choosing a different basis for measurement will impact the outcomes! A measurement made in any qubit state will only yield one of the two states in the measurement basis, each with a certain probability that depends on the state.

When a qubit is measured in the  $\{|0\rangle, |1\rangle\}$  basis, one will extract a result of either 0 or 1; after the measurement, the state of the qubit will be  $|0\rangle$  if 0 was measured, or  $|1\rangle$  if 1 was measured. When a system of many qubits is measured, one gets  $|0\rangle$  or  $|1\rangle$  for each qubit. Thus, a set of  $N$  qubits can simultaneously be in a superposition of  $2^N$  states during quantum computation, but measurement outcomes will yield a set of  $N$  zeroes and ones (either 0 or 1 with different probability for each of the qubits); i.e.,  $N$  bits of classical information are obtained

### **Answer this:**

*Describe in your own words how quantum measurement is different from classical measurement, using no more than 2-3 sentences.*

## Mini-section 1: Measuring a qubit in the $\{|0\rangle, |1\rangle\}$ basis

*You should be able to:*

- *Write and interpret a qubit state written as a superposition of basis states.*
- *Calculate the probability of measuring an output state given a state right before the measurement.*
- *Describe that measurement of the state of each qubit yields one of the two outcomes in the measurement basis.*

### *Orientation*

#### **What is a superposition?**

The representation of the state of a qubit as  $|q\rangle = a|0\rangle + b|1\rangle$  is a linear **superposition** of two basis states,  $\{|0\rangle, |1\rangle\}$ . In a two-state system such as a qubit, a superposition will in general have two orthonormal components added together, no matter what basis the state is expressed in.

**What does a superposition tell us about quantum measurement outcomes?**

In the following questions, we are concerned with the measurement of a qubit written in the form  $|q\rangle = a|0\rangle + b|1\rangle$ . In the  $\{|0\rangle, |1\rangle\}$  basis, the probability that a measurement in the state  $|q\rangle$  yields the state  $|0\rangle$  is  $|a|^2$ , and the probability that such a measurement yields the state  $|1\rangle$  is  $|b|^2$ . (Remember that when states are normalized,  $|a|^2 + |b|^2 = 1$ .)

*Questions*

41. How many orthonormal basis states are there for one qubit?
- a. Infinitely many
  - b. 1
  - c. 2
  - d. Not enough information

*For questions 16-26, consider only measurements that are made in the  $\{|0\rangle, |1\rangle\}$  basis. This means that we do not need to change the basis (recall that changing the basis is equivalent to changing our choice of orthogonal unit vectors).*

42. A qubit is in the state  $|q\rangle = a|0\rangle + b|1\rangle$ , with  $a, b \neq 0$ . Will a qubit in this state always yield the same outcome (i.e., always  $|0\rangle$  or always  $|1\rangle$ ) when measured in the  $\{|0\rangle, |1\rangle\}$  basis?
43. How many possible physically different states are there for a qubit, i.e., for  $|q\rangle = a|0\rangle + b|1\rangle$ , how many possible pairs of  $a$  and  $b$  are possible that would give physically different states?
- a. 1
  - b. 2
  - c. A finite number more than 2
  - d. Infinitely many
44. How many possible states can be obtained when a qubit is measured in a particular basis, e.g., the  $\{|0\rangle, |1\rangle\}$  basis?
- a. 1
  - b. 2
  - c. A finite number more than 2
  - d. Infinitely many
45. Consider a qubit in the state  $|q\rangle = \frac{1}{2}|0\rangle + \frac{i\sqrt{3}}{2}|1\rangle$ . A measurement is performed in the  $\{|0\rangle, |1\rangle\}$  basis.

- (A) What is the probability that the measurement will yield the state  $|0\rangle$ ?

- (B) What is the probability that the measurement will yield the state  $|1\rangle$ ?
- (C) What is the sum of the two probabilities in part (A) and (B)? (Hint: Is the state normalized?)
46. In each case below, for a qubit in the state  $|q\rangle = a|0\rangle + b|1\rangle$ , write down one set of possible values of  $a$  and  $b$  such that the system gives the listed outcomes when a measurement is made in the  $\{|0\rangle, |1\rangle\}$  basis.
- (A) Always yields  $|0\rangle$
- (B) Always yields  $|1\rangle$
- (C) Yields  $|0\rangle$  with 50% probability and  $|1\rangle$  with 50% probability
- (D) Yields  $|0\rangle$  with 36% probability and  $|1\rangle$  with 64% probability
47. Consider a system in the state  $|q\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ . Choose all of the following statements that are true about measurements made in the  $\{|0\rangle, |1\rangle\}$  basis:
- (I) A single measurement will yield  $|q\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$  with 100% probability.
- (II) A single measurement will yield either  $|0\rangle$  or  $|1\rangle$ .
- (III) If the same measurement is performed on a large number of identically-prepared systems, around 36% of the measurements will yield  $|0\rangle$ , and around 64% of the measurements will yield  $|1\rangle$
48. Choose all of the following statements that are correct if measurement is conducted in the  $\{|0\rangle, |1\rangle\}$  basis:
- (I) A bit can only exclusively be in the 0 state or the 1 state. However, a qubit can be in a superposition of  $|0\rangle$  and  $|1\rangle$  states.
- (II) A measurement on a qubit in a given state  $|q\rangle$  can, in general, yield one of two possible outcomes ( $|0\rangle$  or  $|1\rangle$ ).
- (III) When we measure a qubit in a given state, we obtain a single bit of information.

<i>Checkpoints</i>
<ul style="list-style-type: none"> <li>A bit can only exclusively be in the 0 state or the 1 state, while a qubit can be in one of infinitely many superpositions of <math> 0\rangle</math> and <math> 1\rangle</math> states (for example, in the state <math>a 0\rangle + b 1\rangle</math>), one can generate infinitely many physically different states by choosing different</li> </ul>

combinations of  $a$  and  $b$ ). However, there are only two possible measurement outcomes (e.g.,  $|0\rangle$  and  $|1\rangle$  if  $\{|0\rangle, |1\rangle\}$  is chosen as the measurement basis).

- This means that changing the values of  $a$  and  $b$  changes only the probabilities of measuring each outcome.
- When a state is written in a particular basis, e.g., the  $\{|0\rangle, |1\rangle\}$  basis, the expansion coefficients  $a$  and  $b$  are, in general, complex numbers. To find the probability of measurement outcomes, calculate the squared modulus  $|a|^2$  of the complex number by multiplying it by its complex conjugate:  $a \times a^* = |a|^2$
- Each possible outcome is measured with a certain probability in a given quantum state at the end of quantum computation. These probabilities must be determined by making measurements in a large number of identically-prepared systems, rather than making a large number of repeated measurements on a single system. This is because after a single measurement, the quantum state will change and affect the results of future measurements.

### **Additional insight: Consecutive measurements**

*Instructors in disciplines other than physics (e.g., chemistry, quantum information science) may choose to omit this section.*

*You should be able to:*

- *Identify that measurement collapses a state to one of the measurement basis states.*
- *Describe the outcomes of measurements made in immediate succession in a state after it has collapsed from an earlier measurement.*

#### *Orientation*

#### **What happens when a qubit is measured multiple times in immediate succession?**

When measured, a qubit yields only one of the two states in the measurement basis. In addition, measurement can irreversibly *change* the state of the qubit through a process known as **collapse**. For example, if the measurement basis for one qubit is  $\{|0\rangle, |1\rangle\}$ , then measuring 0 means that the qubit has collapsed into the  $|0\rangle$  state, and measuring 1 means it has collapsed into the  $|1\rangle$  state. The original state is destroyed. Measurement does not change a quantum state only if it was already one of the measurement basis states.

#### **Why do we need an ensemble of identically-prepared systems?**

At the end of a quantum computation, we often speak of making a large number of measurements on an ensemble of systems prepared in the same final state. If we tried making multiple measurements in a particular quantum state (often involving multiple qubits), the result of the first measurement would be the same as the result of all subsequent measurements. Therefore, if we wanted to know what outcomes were possible for our measurement, and with what probabilities, we must make a large number of measurements on a system that has been carefully prepared in the desired state prior to each measurement.

#### **Why is measurement collapse important for quantum computing?**

To obtain information from a quantum computer, one ultimately must perform a measurement in the final state of the multi-qubit system, which has been evolved during the computation. It is in this measurement step that the quantum state collapses.

To make connections with the language used in quantum mechanics, measuring a qubit amounts to measuring an observable in a given qubit state. In quantum computing, one often talks about measuring a qubit in a measurement basis instead of measuring an observable. However, choosing a measurement basis is equivalent to performing a measurement of an observable whose eigenstates form the measurement basis. The measurement basis is then composed of the eigenstates of the operator corresponding to that observable. The measurement yields one of the eigenvalues of that operator and collapses the state of a qubit into the eigenstate associated with that eigenvalue. In quantum information science (e.g., quantum computing), if measurement takes place in the standard (or computational) basis, the eigenvalues are often mapped to 0 and 1, and their associated eigenstates represented as  $|0\rangle$  and  $|1\rangle$  respectively.

For the following questions, consider only measurements that are made in the  $\{|0\rangle, |1\rangle\}$  basis.

49. Consider a qubit in the state  $|q\rangle = |0\rangle$ . Choose all of the following statements that are true, assuming all measurements are made in the  $\{|0\rangle, |1\rangle\}$  basis.

- (I) A measurement in this state can yield  $|0\rangle$  or  $|1\rangle$ .
- (II) The outcome of a measurement made on this system can be predicted with 100% certainty.
- (III) After the measurement, the qubit will be in the state  $|0\rangle$ .

50. Consider a system in the state  $|q\rangle = \frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$ . Choose all of the following statements that are true, assuming all measurements are made in the  $\{|0\rangle, |1\rangle\}$  basis.

- (I) A measurement in this state can yield  $|0\rangle$  or  $|1\rangle$ .
- (II) The outcome of a measurement made on this system will be  $|0\rangle$  with 100% probability.
- (III) After a measurement, the system will continue to be in the state  $\frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$ , since the measurement is made in the  $\{|0\rangle, |1\rangle\}$  basis.

51. Consider a qubit in the state  $|q\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ . Choose all of the following statements that are true about measurements made in the  $\{|0\rangle, |1\rangle\}$  basis:

- (I) A measurement in this state will have a 50% probability of yielding  $|0\rangle$  or  $|1\rangle$ , and after this measurement, another one made in immediate succession will also independently have a 50% probability of yielding  $|0\rangle$  or  $|1\rangle$ .
- (II) The first measurement in this state will have a 50% probability of yielding  $|0\rangle$  or  $|1\rangle$ , but a measurement made in immediate succession will yield the same state as the first measurement.

- (III) A measurement in this state will have a 50% probability of yielding  $|0\rangle$  or  $|1\rangle$ , and so do all measurements made on other qubits with identically prepared states.

52. Consider the state  $|q\rangle = \sqrt{\frac{3}{10}}|0\rangle + \sqrt{\frac{7}{10}}|1\rangle$ . A measurement is performed in this basis and yields a result of  $|1\rangle$ . What is the normalized state of the qubit immediately after the measurement?

- $\sqrt{\frac{7}{10}}|1\rangle$
- $\sqrt{\frac{3}{10}}|0\rangle + \sqrt{\frac{7}{10}}|1\rangle$
- $|0\rangle$
- $|1\rangle$
- Not enough information

#### Checkpoints

- Measuring a qubit in the  $\{|0\rangle, |1\rangle\}$  basis yields either the state  $|0\rangle$  or the state  $|1\rangle$ .
- After the measurement, the system is in the state that was obtained. If the measurement yielded  $|0\rangle$ , then after the measurement, the system is in the state  $|0\rangle$ . Conversely, if the measurement yielded  $|1\rangle$ , then the system is in the state  $|1\rangle$ .
- All measurements made in immediate succession will yield the same state as the first measurement.
- To obtain measurement probabilities, measurements are carried out at the end of computation on an ensemble of identically prepared systems. This is accomplished by running the computation multiple times from the same starting state.

## Mini-section 2: Measurements in other bases

You should be able to:

- Find bra states to corresponding ket states (transpose complex conjugates).
- Calculate inner products (specifically inner products related to probabilities of measurement outcomes).
- Explain why changing the measurement basis changes the possible outcomes/probabilities that the measurement will yield those outcomes.
- Identify and describe the properties of orthonormal bases (e.g.,  $\{|0\rangle, |1\rangle\}$ ,  $\{|+\rangle, |-\rangle\}$ ,  $\{|+i\rangle, |-i\rangle\}$ ) and recognize that there are infinitely many orthonormal bases.

#### Orientation

**What is a measurement basis, and what does it mean to make measurements in different bases?**

Here we consider only projective measurements, which can be understood as taking projections of a given vector (the state of the qubit) along a set of orthogonal unit vectors (the measurement basis vectors). A vector has a set magnitude and direction, but a description of the vector can look quite different if one decides to change the basis! Since a measurement can only ever yield the states that serve as the measurement basis states, *changing the measurement basis changes the possible measurement outcomes*.

### Special operations: Inner products

For each ket state  $|q\rangle$ , there is a corresponding bra state  $\langle q|$ . A bra state replaces all numbers with their complex conjugates and reverses the ket brackets,  $|\ \rangle \rightarrow \langle \ |$ . If  $|q\rangle = a|0\rangle + b|1\rangle$ , then  $\langle q| = a^*\langle 0| + b^*\langle 1|$ .

The  $\{|0\rangle, |1\rangle\}$  basis is an orthonormal basis. That means that when the bra and ket “match” and have the same label, their inner product is 1; and when they don’t, the inner product is 0:

$$\begin{aligned}\langle 0|0\rangle &= \langle 1|1\rangle = 1 \\ \langle 0|1\rangle &= \langle 1|0\rangle = 0\end{aligned}$$

We can take a moment to appreciate some “special” states. What happens when, in a state  $|q\rangle = a|0\rangle + b|1\rangle$ ,  $|a| = |b|$ ?

This gives us some states for  $|q\rangle$  that can have their own labels. Specifically, if  $\{|0\rangle, |1\rangle\}$  represent the eigenstates of  $\sigma_z$ , we have

$$\begin{aligned}|+\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\ |-\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\end{aligned}$$

(The above are the eigenstates of  $\sigma_x$ .)

$$\begin{aligned}|+i\rangle &= \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \\ |-i\rangle &= \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle\end{aligned}$$

(The above are the eigenstates of  $\sigma_y$ .)

### Questions

53. Write down the bra states (in the  $\{|0\rangle, |1\rangle\}$  basis) that correspond to each of the following ket states.

(A)  $|a\rangle = |0\rangle$

(B)  $|b\rangle = |1\rangle$

$$(C) \quad |c\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

$$(D) \quad |d\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |+i\rangle$$

$$(E) \quad |e\rangle = \frac{5}{13}|0\rangle + \frac{12}{13}|1\rangle$$

$$(F) \quad |f\rangle = \sqrt{\frac{7}{10}}|0\rangle - \sqrt{\frac{3}{10}}i|1\rangle$$

$$(G) \quad |g\rangle = \frac{1+i}{\sqrt{15}}|0\rangle + \frac{2-3i}{\sqrt{15}}|1\rangle$$

54. Calculate the following inner products, using the states  $|a\rangle$  through  $|g\rangle$  from the previous question. Don't forget the following identities:

$$\langle 0|0\rangle = 1 \qquad \langle 0|1\rangle = 0$$

$$\langle 1|1\rangle = 1 \qquad \langle 1|0\rangle = 0$$

(A)  $\langle c|e\rangle$

(B)  $\langle e|c\rangle$

(C)  $\langle d|g\rangle$

(D)  $\langle g|d\rangle$

<i>Checkpoints</i>
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- |  |
|--|
| <ul style="list-style-type: none"> <li>For any inner product <math>\langle p q\rangle</math>, reversing the ket and bra results in its complex conjugate: <math>\langle p q\rangle = \langle q p\rangle^*</math>.</li> </ul> |
|--|

55. If two states  $|q\rangle$  and  $|p\rangle$  form an orthonormal basis, then  $\langle q|q\rangle = \langle p|p\rangle = 1$  and  $\langle q|p\rangle = \langle p|q\rangle = 0$ .

Do  $|+\rangle$  and  $|-\rangle$  form an orthonormal basis? A straightforward way to check is to do these inner products in the  $\{|0\rangle, |1\rangle\}$  basis. (Refer to the *Orientation* box for how to express the states  $|+\rangle$  and  $|-\rangle$  in the  $\{|0\rangle, |1\rangle\}$  basis.)

56. Do  $|+i\rangle$  and  $| -i\rangle$  form an orthonormal basis?

57. How many orthonormal basis sets exist in a 2-D vector space?

- a. 1
- b. The 3 found above
- c. More than 1 but less than infinity
- d. Infinitely many
- e. None of the above

Consider a state  $|q\rangle = a|0\rangle + b|1\rangle$ . For questions 32-34, choose all of the correct options. (There may or may not be more than one correct option.)

58. If we make a measurement in the state  $|q\rangle$  in the  $\{|0\rangle, |1\rangle\}$  basis, what are possible outcomes of the measurement?

- (I)  $|0\rangle$  or  $|1\rangle$
- (II)  $|+\rangle$  or  $|-\rangle$
- (III)  $|+i\rangle$  or  $|-i\rangle$

59. If we make a measurement in the state  $|q\rangle$  in the  $\{|+\rangle, |-\rangle\}$  basis, what are possible outcomes of the measurement?

- (I)  $|0\rangle$  or  $|1\rangle$
- (II)  $|+\rangle$  or  $|-\rangle$
- (III)  $|+i\rangle$  or  $|-i\rangle$

60. If we make a measurement in the state  $|q\rangle$  in the  $\{|+i\rangle, |-i\rangle\}$  basis, what are possible outcomes of the measurement?

- (I)  $|0\rangle$  or  $|1\rangle$
- (II)  $|+\rangle$  or  $|-\rangle$
- (III)  $|+i\rangle$  or  $|-i\rangle$

61. If we perform a measurement in each of the following states in the  $\{|0\rangle, |1\rangle\}$  basis, what are the outcomes, and probabilities of measuring those outcomes?

Refer to the *Orientation* box to write each state in the  $\{|0\rangle, |1\rangle\}$  basis (i.e., in the form  $a|0\rangle + b|1\rangle$ ), and use what you know about the probabilities of measurement outcomes.

- (A)  $|0\rangle$
- (B)  $|1\rangle$
- (C)  $|+\rangle$
- (D)  $|-\rangle$
- (E)  $|+i\rangle$
- (F)  $|-i\rangle$

62. Recall that the states  $|+\rangle$  and  $|-\rangle$  are expressed in the  $\{|0\rangle, |1\rangle\}$  basis as follows:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Now express  $|0\rangle$  and  $|1\rangle$  in the  $\{|+\rangle, |-\rangle\}$  basis, i.e., in terms of the states  $|+\rangle$  and  $|-\rangle$ . Add (or subtract) the states  $|+\rangle$  and  $|-\rangle$ , and isolate  $|0\rangle$  or  $|1\rangle$  to get expressions for them.

63. The states  $|+i\rangle$  and  $| -i\rangle$  can be expressed in the  $\{|+\rangle, |-\rangle\}$  basis as follows:

$$|+i\rangle = e^{\frac{i\pi}{4}} \left( \frac{1}{\sqrt{2}}|+\rangle - \frac{i}{\sqrt{2}}|-\rangle \right)$$

$$|-i\rangle = e^{-\frac{i\pi}{4}} \left( \frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle \right)$$

You may notice that these relations look similar to the states  $|+i\rangle$  and  $| -i\rangle$  as they are expressed in the  $\{|0\rangle, |1\rangle\}$  basis! Using this knowledge, and your results from the preceding question, answer this question:

If we perform a measurement in each of the following states in the  $\{|+\rangle, |-\rangle\}$  basis, what are the outcomes, and probabilities of measuring those outcomes?

- (A)  $|+\rangle$
- (B)  $|-\rangle$
- (C)  $|+i\rangle$
- (D)  $| -i\rangle$
- (E)  $|0\rangle$
- (F)  $|1\rangle$

#### Checkpoints

- **In the  $\{|0\rangle, |1\rangle\}$  basis**, measuring a qubit in the state  $|0\rangle$  will yield  $|0\rangle$  with 100% certainty, and measuring a qubit in the state  $|1\rangle$  will yield  $|1\rangle$  with 100% certainty. However, measuring a qubit in any of the states  $|+\rangle, |-\rangle, |+i\rangle$ , and  $| -i\rangle$  will yield either  $|0\rangle$  or  $|1\rangle$  with 50% probability.
- **In the  $\{|+\rangle, |-\rangle\}$  basis**, the states  $|+\rangle$  and  $|-\rangle$  will yield outcomes with 100% certainty, but  $|0\rangle, |1\rangle, |+i\rangle$ , and  $| -i\rangle$  will yield either  $|+\rangle$  or  $|-\rangle$  with 50% probability.

- **In the  $\{|+i\rangle, |-i\rangle\}$  basis**, the states  $|+i\rangle$  and  $|-i\rangle$  will yield outcomes with 100% certainty, but  $|0\rangle, |1\rangle, |+\rangle$ , and  $|-\rangle$  will yield either  $|+i\rangle$  or  $|-i\rangle$  with 50% probability.
- Note: The following states are expressed in the  $\{|+i\rangle, |-i\rangle\}$  basis.
- $|0\rangle = \frac{1}{\sqrt{2}}|+i\rangle + \frac{1}{\sqrt{2}}|-i\rangle$
- $|1\rangle = e^{-\frac{i\pi}{2}} \left( \frac{1}{\sqrt{2}}|+i\rangle - \frac{1}{\sqrt{2}}|-i\rangle \right)$
- $|+\rangle = e^{-\frac{i\pi}{4}} \left( \frac{1}{\sqrt{2}}|+i\rangle + \frac{i}{\sqrt{2}}|-i\rangle \right)$
- $|-\rangle = e^{\frac{i\pi}{4}} \left( \frac{1}{\sqrt{2}}|+i\rangle - \frac{i}{\sqrt{2}}|-i\rangle \right)$

64. Consider a qubit in the state  $|q\rangle = \frac{3}{5}|0\rangle - \frac{4i}{5}|1\rangle$ .

(A) Calculate  $\langle 0|q\rangle$  and  $\langle 1|q\rangle$  explicitly by writing  $|q\rangle$  as the given superposition and carrying out the inner products.

(B) How are  $\langle 0|q\rangle$  and  $\langle 1|q\rangle$  related to  $a$  and  $b$  for the state written as  $|q\rangle = a|0\rangle + b|1\rangle$ ?

How would you use  $\langle 0|q\rangle$  and  $\langle 1|q\rangle$  to find the respective probabilities that a measurement made on  $|q\rangle$  will yield  $|0\rangle$  or  $|1\rangle$ ?

#### Checkpoints

- There are 2 methods of calculating probabilities of measurement outcomes. The first is simply to write the state in the measurement basis, for example  $|q\rangle = a|0\rangle + b|1\rangle$  in the  $\{|0\rangle, |1\rangle\}$  basis, and then take absolute square of each coefficient  $|a|^2$  and  $|b|^2$ .
- The second method is to take the absolute square of the inner products  $\langle 0|q\rangle$  and  $\langle 1|q\rangle$ . If the measurement basis is identical to the computational basis (or standard basis), the methods are equivalent.
- Remember that, since  $\langle a|q\rangle = \langle q|a\rangle^*$ , this means that, regardless of whether we calculate the probability of yielding the state  $|a\rangle$  given state  $|q\rangle$  or  $|q\rangle$  given state  $|a\rangle$ , the probabilities  $|\langle a|q\rangle|^2 = |\langle q|a\rangle|^2$  are the same!

65. Consider a system in the state  $|d\rangle = \frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$  and answer the following:

(A) Does there exist some measurement basis in which the measurement will not change the state  $|d\rangle = \frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$  (i.e., the state will remain the same before and after the measurement)?

*Checkpoint*

The basis states will be  $|d\rangle = \frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$  and  $|-d\rangle = \sqrt{\frac{6}{7}}|0\rangle - \frac{1}{\sqrt{7}}|1\rangle$ .

(B) If we make a measurement in the state  $|d\rangle$  in the  $\{|d\rangle, |-d\rangle\}$  basis, can the outcome of the measurement be predicted with 100% certainty? Explain.

(C) If we make a measurement in the state  $|0\rangle$  in the  $\{|d\rangle, |-d\rangle\}$  basis, can the outcome of the measurement be predicted with 100% certainty? Explain.

(Hint: What are the possible outcomes? Calculate the respective probabilities using inner products. Note that if you can find  $\langle 0|d\rangle$ , calculating  $|\langle d|0\rangle|^2$  is straightforward because  $|\langle d|0\rangle|^2 = |\langle 0|d\rangle|^2$ .)

(D) If we make a measurement in the state  $|1\rangle$  in the  $\{|d\rangle, |-d\rangle\}$  basis, what are the possible outcomes of the measurement, and what are the probabilities of those outcomes?

(E) Consider a state  $|b\rangle = \frac{2}{\sqrt{10}}|0\rangle + \sqrt{\frac{6}{10}}|1\rangle$ . If we make a measurement in the state  $|b\rangle$  in the  $\{|d\rangle, |-d\rangle\}$  basis, what is the probability of yielding the state  $|d\rangle$ ?

(Hint: Is it easier to express  $|b\rangle$  as  $a|d\rangle + b|-d\rangle$ , or to find the inner product  $\langle b|d\rangle$  in the  $\{|0\rangle, |1\rangle\}$  basis?)

*Checkpoints (collected)*

- For any inner product  $\langle p|q\rangle$ , reversing the ket and bra results in its complex conjugate:  $\langle p|q\rangle = \langle q|p\rangle^*$ .
- The states  $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |+i\rangle$ , and  $|-i\rangle$  are constructed such that when measured in any of the bases  $\{|0\rangle, |1\rangle\}$ ,  $\{|+\rangle, |-\rangle\}$ , and  $\{|+i\rangle, |-i\rangle\}$ , any of these states that is not already a basis state will yield each possible outcome with exactly 50% probability.
- One way to calculate the probabilities of measurement outcomes is to write the state in the measurement basis, for example  $|q\rangle = a|0\rangle + b|1\rangle$  in the  $\{|0\rangle, |1\rangle\}$  basis, and then take the absolute square of each coefficient  $|a|^2$  and  $|b|^2$ .
- Another way is to take the absolute square of the inner products  $\langle 0|q\rangle$  and  $\langle 1|q\rangle$ .
- For any given state, one can find a state that is orthonormal to it, and use the two states to form an orthonormal basis.

### Mini-section 3: Changing the basis in 2-D Hilbert space

You should be able to:

- Change a state represented in one basis, for example the  $\{|0\rangle, |1\rangle\}$  basis, to a representation in another basis, such as the  $\{|+\rangle, |-\rangle\}$  and  $\{|+i\rangle, |-i\rangle\}$  bases.
- Use the spectral decomposition of the identity operator to change basis.

*Orientation*

#### **Why is it important to change the basis?**

Each measurement of a qubit has two basis states associated with it, one for each possible outcome. However, if a given state is not already written in terms of the measurement basis, then it can be difficult to calculate the probability of obtaining each outcome without first transforming into the measurement basis. Again, when considering projective measurements, an inner product along each basis vector can be considered a projection of the state along that basis vector.

In the standard (computational) basis of a quantum computer, each qubit is expressed in the  $\{|0\rangle, |1\rangle\}$  basis. However, the measurement basis can be this basis, or some other basis such as the  $\{|+\rangle, |-\rangle\}$  basis—or even a special basis known as the Bell basis, which uses entangled states as basis states. However, measurements can be made in the standard basis (even if measurement in the  $\{|+\rangle, |-\rangle\}$  basis is desired) by first performing the appropriate basis change through operations with quantum gates before the measurement.

*Questions*

66. Given that  $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$  and  $|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$ , write each of the following states using the basis states  $|+\rangle$  and  $|-\rangle$ .

(A)  $a|0\rangle$

(B)  $b|1\rangle$

(C)  $|q\rangle = a|0\rangle + b|1\rangle$

(D)  $\frac{2}{5}|0\rangle + \sqrt{\frac{21}{25}}|1\rangle$

### Checkpoints

- To convert any state represented in the  $\{|0\rangle, |1\rangle\}$  basis to the  $\{|+\rangle, |-\rangle\}$  basis, use this relation:  $|q\rangle = a|0\rangle + b|1\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$
- To convert any state represented in the  $\{|0\rangle, |1\rangle\}$  basis to the  $\{|+i\rangle, |-i\rangle\}$  basis, use this relation:  $|q\rangle = a|0\rangle + b|1\rangle = \frac{a-ib}{\sqrt{2}}|+i\rangle + \frac{a+ib}{\sqrt{2}}|-i\rangle$  (It is possible, but not required, to perform the same procedure to convince yourself that this is so.)

67. Consider again the state  $|q\rangle = a|0\rangle + b|1\rangle$ . Calculate  $\langle +|q\rangle$  using the following methods:

(A) Write both  $|q\rangle$  and  $\langle +|$  in the  $\{|0\rangle, |1\rangle\}$  basis to calculate the inner product.

(B) Write both  $|q\rangle$  and  $\langle +|$  in the  $\{|+\rangle, |-\rangle\}$  basis to calculate the inner product. (Hint:  $|q\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$ .)

(C) Do your answers to (A) and (B) match? Explain.

(D) Explain what  $\langle +|q\rangle$  represents, in words.

68. Now calculate  $\langle -|q\rangle$ .

(A) Write both  $|q\rangle$  and  $\langle -|$  in the  $\{|0\rangle, |1\rangle\}$  basis to calculate the inner product.

(B) Write both  $|q\rangle$  and  $\langle -|$  in the  $\{|+\rangle, |-\rangle\}$  basis to calculate the inner product.

(C) Do your answers to (A) and (B) match? Explain.

(D) Explain what  $\langle -|q\rangle$  represents, in words.

(E) Add  $|\langle +|q\rangle|^2$  and  $|\langle -|q\rangle|^2$ . What do you get? Does it make sense? Explain.

(F) Write  $\langle +|q\rangle\langle +| + \langle -|q\rangle\langle -|$ , substituting  $\langle +|q\rangle$  and  $\langle -|q\rangle$  with the answers that you found. Does it look familiar? What does this represent?

### Checkpoints

- The inner product  $\langle p|q\rangle$  is the probability amplitude of measuring the state  $|p\rangle$  as an outcome, when the measurement is made in the state  $|q\rangle$ . ( $|p\rangle$  must be one of the states of the measurement basis for this interpretation to hold. For the purposes of the inner product, the bra state  $\langle p|$  must be used.) [This is known as the *Born rule*.]
- Another way to think about the inner product is that it is the projection of the ket state onto the bra state. For example,  $\langle +|q\rangle$  can be thought of as the projection of the state  $|q\rangle$  onto the state  $|+\rangle$ .
- Adding together  $\langle +|q\rangle|+\rangle$  and  $\langle -|q\rangle|-\rangle$  in superposition fully expresses the state  $|q\rangle$  in the  $\{|+\rangle, |-\rangle\}$  basis. [This is known as the *spectral resolution of unity*.]

### Special operations: Outer products

An outer product is an operator and is constructed by placing a ket before a bra:  $|ket\rangle\langle bra|$ . Each one can be thought of as a matrix element. Outer products such as  $|0\rangle\langle 0|$ ,  $|0\rangle\langle 1|$ ,  $|1\rangle\langle 0|$ , and  $|1\rangle\langle 1|$  are operators and, unlike inner products, not numbers such as 0 or 1. Quantum gates can be expressed as linear combinations of outer products.

For the following questions 43-45, suppose we have a ket state  $|q\rangle = a|0\rangle + b|1\rangle$  and a bra state  $\langle p| = c\langle 0| + d\langle 1|$ . We will contrast the inner product with the outer product.

69. First, calculate the inner product  $\langle p|q\rangle$ .

(A) Substitute  $\langle p|$  and  $|q\rangle$  with the appropriate superposition states in the  $\{|0\rangle, |1\rangle\}$  basis and carry out the inner products. Before eliminating anything, you should have four terms.

(B) What are  $\langle 0|0\rangle$ ,  $\langle 0|1\rangle$ ,  $\langle 1|0\rangle$ , and  $\langle 1|1\rangle$ ? (Hint: remember that  $|0\rangle, |1\rangle$  form an orthonormal basis.) Replace these terms. Simplify.

70. Now, calculate the outer product  $|q\rangle\langle p|$ .

(A) Again, substitute  $\langle p|$  and  $|q\rangle$  with the appropriate superposition states and carry out the outer products. You should again get four terms.

Order matters:  $|0\rangle\langle 0| \neq \langle 0|0\rangle$ ! Leave these terms the way they are.

(B) Compare to your answer for the last problem. Do you see any similarities or differences? Explain.

71. Select all of the following statements that are correct:

(II)  $|0\rangle\langle 0| = 0$

(III)  $|0\rangle\langle 0| = |1\rangle\langle 1| = 1$

(IV)  $|0\rangle\langle 1| = |1\rangle\langle 0| = 0$

*Checkpoint*

- Inner products of orthonormal basis states are either zero or one, but outer products of orthonormal basis states are *neither* zero nor one! They are in fact not numbers at all but rather operators, which can be represented as  $2 \times 2$  matrices in a given basis.

**Bottom line: Never simplify  $|0\rangle\langle 0|$  or  $|1\rangle\langle 1|$  to 1, and never simplify  $|0\rangle\langle 1|$  or  $|1\rangle\langle 0|$  to 0.**

*For the following questions:*

- $|q\rangle = a|0\rangle + b|1\rangle$  and  $|p\rangle = c|0\rangle + id|1\rangle$ , as given earlier. Substitute each instance of  $|p\rangle$  and  $|q\rangle$  and their respective ket and bra states with the given superposition states.
- The parentheses written out explicitly in parts (A)-(C) are not important, since the associative property holds when bras and kets are multiplied by one another.

72. Though inner products are simply numbers (because the inner products of the basis states are either zero or one), the outer products can be multiplied with bras or kets to simplify to other mathematical entities. Calculate the following products:

(A)  $(|0\rangle\langle 0|)(|0\rangle) = |0\rangle\langle 0|0\rangle$

(B)  $(|0\rangle\langle 0|)(|1\rangle) = |0\rangle\langle 0|1\rangle$

(C)  $(\langle 1|)(|1\rangle\langle 0|) = \langle 1|1\rangle\langle 0|$

(D)  $|p\rangle\langle 0|q\rangle$

(E)  $(|0\rangle\langle 0|)(|1\rangle\langle 1|) = |0\rangle\langle 0|1\rangle\langle 1|$

(F)  $(|0\rangle\langle 1|)(|1\rangle\langle 0|) = |0\rangle\langle 1|1\rangle\langle 0|$

(G)  $\langle 1|q\rangle\langle p|0\rangle$

73. The identity operator  $\hat{I}$  is an operator that can act on a state without changing it. It is analogous to multiplying a number by 1. The identity operator can be represented in *any* orthonormal basis by taking the outer product of each orthonormal basis vector with itself, and then summing together all of these outer products.

(A) In the  $\{|0\rangle, |1\rangle\}$  basis,  $\hat{I} = |0\rangle\langle 0| + |1\rangle\langle 1|$ .

Using  $|q\rangle = a|0\rangle + b|1\rangle$ , verify that  $\hat{I}|q\rangle = |q\rangle$  by replacing  $\hat{I}$  with  $|0\rangle\langle 0| + |1\rangle\langle 1|$  and carrying out the inner and outer products.

(B) In the  $\{|+\rangle, |-\rangle\}$  basis,  $\hat{I} = |+\rangle\langle +| + |-\rangle\langle -|$ . Compute  $\hat{I}|q\rangle$  again, this time in the  $\{|+\rangle, |-\rangle\}$  basis.

(C) Recall that  $|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$  and  $|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$ . Calculate the following inner products:

a.  $\langle +|0\rangle$

b.  $\langle +|1\rangle$

c.  $\langle -|0\rangle$

d.  $\langle -|1\rangle$

Simplify your answer to (B) with these results. Does your result still represent the same state  $|q\rangle$ ? Compare your answer to that of question 40, part (C). (The answer is given in the first *Checkpoint* box in this mini-section.)

(D) In the  $\{|+i\rangle, |-i\rangle\}$  basis,  $\hat{I} = |+i\rangle\langle +i| + |-i\rangle\langle -i|$ .

Compute  $\hat{I}|q\rangle$  in the  $\{|+i\rangle, |-i\rangle\}$  basis, and simplify your answer using the following inner products (which are obtained in the same manner):

$$\langle +i|0\rangle = \frac{1}{\sqrt{2}} \quad \langle +i|1\rangle = -\frac{i}{\sqrt{2}} \quad \langle -i|0\rangle = \frac{1}{\sqrt{2}} \quad \langle -i|1\rangle = \frac{i}{\sqrt{2}}$$

Compare your answer to that given in the first *Checkpoint* box in this mini-section.

74. Express  $|q\rangle$  in the  $\{|d\rangle, |-d\rangle\}$  basis.

(A) Using  $|d\rangle = \frac{1}{\sqrt{7}}|0\rangle + \sqrt{\frac{6}{7}}|1\rangle$  and  $|-d\rangle = \sqrt{\frac{6}{7}}|0\rangle - \frac{1}{\sqrt{7}}|1\rangle$ , first express  $|0\rangle$  and  $|1\rangle$  in terms of the states  $|d\rangle$  and  $|-d\rangle$ , and then use these to represent  $|q\rangle = a|0\rangle + b|1\rangle$  in the  $\{|d\rangle, |-d\rangle\}$  basis.

(B) Express  $\hat{I}$  in the  $\{|d\rangle, |-d\rangle\}$  basis and then compute  $\hat{I}(a|0\rangle + b|1\rangle)$  by carrying out the inner and outer products. Compare your answer to that of (A).

#### Checkpoints

- If  $|q\rangle = a|0\rangle + b|1\rangle$  in the  $\{|0\rangle, |1\rangle\}$  basis, then  $|q\rangle = \frac{a+b}{\sqrt{2}}|+\rangle + \frac{a-b}{\sqrt{2}}|-\rangle$  in the  $\{|+\rangle, |-\rangle\}$  basis and  $|q\rangle = a|0\rangle + b|1\rangle = \frac{a-ib}{\sqrt{2}}|+i\rangle + \frac{a+ib}{\sqrt{2}}|-i\rangle$  in the  $\{|+i\rangle, |-i\rangle\}$  basis.
- The inner product  $\langle p|q\rangle$  can be thought of as the probability amplitude of measuring the bra state  $\langle p|$  in the given ket state  $|q\rangle$ , or as the projection of the state  $|q\rangle$  onto the state  $|p\rangle$ .
- The state  $|q\rangle$  can also be expressed as  $|q\rangle = \langle +|q\rangle|+\rangle$  and  $\langle -|q\rangle|-\rangle$  in the  $\{|+\rangle, |-\rangle\}$  basis.
- Outer products behave differently from inner products. Never simplify  $|0\rangle\langle 0|$  or  $|1\rangle\langle 1|$  to 1, and never simplify  $|0\rangle\langle 1|$  or  $|1\rangle\langle 0|$  to 0.
- Another way to change the basis is to multiply the state by the identity operator expressed in the destination basis. This preserves the state, but expresses it in a different way.

#### Summary

A change in basis is sometimes desirable or necessary when working with quantum states of qubits. For example, the measurement basis may be different from the computational basis (standard basis), the basis in which the state is given. In these cases, there are several ways to perform and understand the process of changing the basis, all of them mathematically equivalent.

## Section 3: Quantum computation

### *Introduction*

A quantum computer that is used to run a computation must go through the steps of initialization, computation, and output.

- The initialization step ensures that the quantum computer starts from a known state each time it is used to run a quantum algorithm.
- Computation is achieved by using quantum gates to transform the state of the quantum computer in various ways based on the desired outcome of the quantum algorithm.
- Once all the quantum gates have been applied, the output is obtained by making a measurement on the final state of the quantum computer. If this state is a superposition of the measurement basis states, then measurement outcomes are random and based upon certain probabilities of obtaining each of the two outcomes for each qubit.

### Mini-section 1: Initialization

*You should be able to:*

- *Identify the number of bits of information that are needed to initialize a quantum computer*
- *Describe the role of initialization in a quantum computation*

*Orientation*

#### **What is initialization?**

Initialization is simply a way of setting up the quantum computer so that the quantum computation can start from a specified state. In theory, the computation should proceed predictably if it starts from the same initial state each time.

75. Consider the following conversation between two students:

**Student 1:** To initialize an  $N$ -bit classical computer, you can set the  $N$  bit values to zero, so you perform  $N$  operations total. But to initialize an  $N$ -qubit quantum computer, you can set  $2^N$  qubit states to  $|0\rangle$ , so you perform  $2^N$  operations.

**Student 2:** Wait, that doesn't make sense. There are only  $N$  qubits, and you only need to know the starting state. You only need to set the value of  $N$  qubits, similar to setting  $N$  bits to zero in the classical computer. You can't initialize  $2^N$  qubits when you only actually have  $N$  qubits.

Which student(s), if any, do you agree with? Explain.

76. Consider the following conversation between several students:

**Student 1:** I think that during the initializing process, you need to set the value of the  $N$  qubits to  $|1\rangle$ , not  $|0\rangle$ .

**Student 2:** Actually, you just need to know what the starting state is for each qubit in order to do computations with them, but it doesn't matter exactly what state they are in. As long as you know exactly which qubits in the string were  $|0\rangle$  or  $|1\rangle$ , you could even initialize them to a mix of both. It's just easiest to set them all to the same value.

**Student 3:** Right! For example, a 3-qubit quantum computer could be initialized to any of the states  $|000\rangle$ ,  $|111\rangle$ , or  $|001\rangle$ , because each of these states is one of eight basis states for a 3-qubit system when written out in the  $\{|0\rangle, |1\rangle\}$  basis for each qubit.

Which student(s), if any, do you agree with? Explain.

77. Consider the following conversation between several students:

**Student 1:** Okay, then if you choose to initialize a 3-qubit quantum computer to, say, the state  $|001\rangle$ , that would be the same as initializing it to  $|010\rangle$  or  $|100\rangle$ .

**Student 2:** No,  $|001\rangle$ ,  $|010\rangle$ , and  $|100\rangle$  are actually three completely distinct states. Even though they have the same numbers of zeroes and ones, the placement of those zeroes and ones is different in each of them. Each of the eight basis states of the 3-qubit system is therefore unique and not interchangeable, but they are all equally valid states to start in.

**Student 3:** I agree with Student 2. Physically, the  $|001\rangle$  state corresponds to the first two qubits being set to  $|0\rangle$  and the third qubit being set to  $|1\rangle$ . On the other hand, the  $|100\rangle$  state corresponds to the first qubit being set to  $|1\rangle$  and the second and third qubits being set to  $|0\rangle$ , so the two states are indeed different.

Which student(s), if any, do you agree with? Explain.

78. Consider the following conversation between two students:

**Student 1:** Can't you write a state of the quantum computer as a superposition of all  $2^N$  states? Then surely you would need to be able to perform  $2^N$  operations, one on each of those  $2^N$  coefficients, in order to initialize the whole quantum computer. For example, the value of the  $|000 \dots 0\rangle$  coefficient could be 1, and the rest could all be 0.

**Student 2:** Actually, the  $|000 \dots 0\rangle$  state you just described is obtained simply by initializing the  $N$  qubits to  $|0\rangle$ . It is equivalent to the state where the coefficient of  $|000 \dots 0\rangle$  is 1 and the other coefficients are 0.

**Student 1:** So you're saying that there are two valid ways of obtaining the initialized state? One is by setting values for each of the  $2^N$  basis state coefficients of an  $N$ -qubit system. And the other is setting values for each of the qubits.

**Student 2:** No, not quite. There is a mathematical way of describing the state of a quantum computer, which is writing down the state with the  $2^N$  coefficients, and a physical way of initializing the quantum computer, e.g., by setting the values of each of the  $N$  qubits to  $|0\rangle$ . We can physically initialize a quantum computer by performing operations on the individual qubits.

Which student(s), if any, do you agree with? Explain.

#### Checkpoints

- There are  $2^N$  linearly independent (mutually exclusive) basis states available to an  $N$ -qubit quantum computer, but each of the  $2^N$  states contributes only  $N$  bits of information. To set each qubit to a definite prespecified state (i.e., to initialize the quantum computer), such as a string of all zeroes or all ones, one only needs to assign a value to each of the  $N$  qubits—*not* all  $2^N$  amplitudes of all the basis states!
- A quantum computer can be initialized to any state that is convenient, as long as the state is known beforehand so that the quantum algorithm to be applied can proceed as expected.

### Mini-section 2: Quantum gates and computations

*You should be able to:*

- *Refute the notion that quantum computations must be kept track of on classical registers*
- *Describe how quantum computation is different from running multiple classical computations in parallel*
- *Describe the difference between superposition and entanglement*
- *Identify the cases in which a qubit can be in 1) a superposition state, 2) an entangled state, and 3) a superposition state which isn't an entangled state*

#### Orientation

We discussed before that quantum gates (i.e., quantum operators) can transform one quantum state into a different quantum state. Quantum gates are unitary: they serve to rotate a quantum state in Hilbert space while preserving its magnitude.

In general, when a quantum computation takes place in an  $N$ -bit quantum computer, a superposition of  $2^N$  linearly independent states evolves in time. After initialization of the qubit states, quantum gates can be used to do computation. Applying these gates to the initial quantum state can turn it into a superposition of linearly independent states in some basis, and if different qubits interact, the state may even be entangled. For instance, the Hadamard gate can turn a specific initial state, e.g., a qubit in the  $|0\rangle$  state or the  $|1\rangle$  state, into an equal superposition of the  $|0\rangle$  and  $|1\rangle$  states. As another example, the CNOT gate [not discussed in this tutorial], which is a two-qubit gate, can turn the initial state of two qubits into an entangled state. For an  $N$ -qubit system:

$$\widehat{U}|000 \dots 00\rangle = a_0|000 \dots 00\rangle + a_1|000 \dots 01\rangle + a_2|000 \dots 10\rangle + \dots + a_{2^N-1}|111 \dots 11\rangle$$

where  $\widehat{U}$  is a (unitary) quantum gate, and

$$|a_0|^2 + |a_1|^2 + |a_2|^2 + \dots + |a_{2^N-1}|^2 = 1$$

Let's look at the Hadamard gate again in detail. For a single qubit, a two-state system, the Hadamard gate looks like  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ . Applying the Hadamard gate to the  $|0\rangle$  state looks like this:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

And to the  $|1\rangle$  state, it looks like this:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

As you can see, the Hadamard gate turns the  $|0\rangle$  or  $|1\rangle$  state into an equal superposition of those states: the  $|+\rangle$  and  $|-\rangle$  states, respectively.

In addition to the Hadamard gate, the  $\sigma_x$  and  $\sigma_z$  gates are also of interest. Let's take a look at how they change the states to which they are applied.

79. Consider the  $\sigma_x$  gate, which looks like this in matrix form:  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(A) Apply the  $\sigma_x$  gate to each of the following states. What do you get? Write your answer in the right column.

$ q\rangle$	$\sigma_x q\rangle$
$ 0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	
$ 1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	
$a 0\rangle + b 1\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix}$	

(B) If you made a large number of measurements on systems identically prepared in  $|q\rangle$  and  $\sigma_x|q\rangle$ , how would the measurement results compare between the two states?

(C) You may remember that  $|+\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$  and  $|-\rangle \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  are the eigenstates of  $\sigma_x$ . What do you get when you apply  $\sigma_x$  to these states?

(D) Explain, in words, what the  $\sigma_x$  gate does to the state  $|0\rangle$  and the state  $|1\rangle$ . (Hint: This is sometimes called a “bit flip”; do you think this name make sense?)

80. Now consider the  $\sigma_z$  gate,  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(A) Apply the  $\sigma_z$  gate to each of the following states. What do you get?

$ q\rangle$	$\sigma_z q\rangle$
$ +\rangle$	
$ -\rangle$	
$a 0\rangle + b 1\rangle$	

(B) The eigenstates of  $\sigma_z$  are  $|0\rangle$  and  $|1\rangle$ . Apply  $\sigma_z$  to each of these states. If you ignore overall phase factors, what do you get in each case?

(C) If you made a large number of measurements on systems identically prepared in  $|q\rangle$  and  $\sigma_z|q\rangle$ , how would the measurement results compare between the two states?

(D) Explain, in words, what the  $\sigma_z$  gate does to the state  $|+\rangle$  and the state  $|-\rangle$ . Do you think the name “phase flip” makes sense? Explain.

81. Consider the following conversation between several students:

**Student 1:** While the quantum computation is taking place on a quantum computer, we need to keep track of each of these  $2^N$  coefficients in some classical way, such as on classical registers.

**Student 2:** Wouldn't that defeat the purpose of trying to build a large quantum computer in the first place? You wouldn't be able to make even 200-qubit quantum computers if we needed to keep track of the coefficients in classical registers. You're basically having a classical computer simulate the 200-qubit quantum computer at that point.

**Student 3:** Well, that's why it's infeasible to make a large quantum computer, because a 1000-qubit quantum computer would require knowing the values of  $2^{1000} \cong 10^{300}$  coefficients, which is greater than the number of all the atoms in the universe!

**Student 4:** I disagree with Students 1 and 3. You never need to keep track of those  $2^N$  coefficients on a quantum computer, because the qubits naturally do it themselves. Moreover, we actually can't even find out the values of those coefficients. Reading the “output” of the

computation involves measurement, which collapses the superposition state to one of the basis states, which is an  $N$ -bit string of classical information. This is much less than the information contained in  $2^N$  coefficients.

Which student(s), if any, do you agree with? Explain.

82. Consider the following conversation between several students:

**Student 1:** Because an  $N$ -qubit quantum computer deals with superpositions of states instead of one state at a time, a quantum computer can run many different instances of the same classical algorithm at once. This reduces computation time, which makes an  $N$ -qubit quantum computer faster than an  $N$ -bit classical computer.

**Student 2:** I think you just described many classical computers being run in parallel. Quantum computers use a very different approach. A quantum computer in a superposition state gives us a single string of  $N$  bits of classical information when the output is measured at the end of a computation. That answer could be right or wrong, so we want the quantum algorithm to make the right answer much more probable than the others. As long as this is accomplished efficiently, the quantum algorithm can be useful.

**Student 3:** I still agree with Student 1. My understanding of Grover's algorithm [not discussed in this tutorial], for example—which is an unsorted search, like if you searched in a phone book ordered by name while only knowing the person's phone number—is that a classical computer can only search the entries one by one, but a quantum computer could search many entries at the same time.

**Student 2:** No, Grover's algorithm is actually very different from that. A special quantum gate (in particular, application of the Hadamard gate to each of the qubits) is used to turn the initialized state into a superposition state with equal probability of measuring each of the  $2^N$  possibilities. If you were to make a measurement at that point, your probability of getting the right answer is not zero, but it's very low. Further applications of other quantum gates will increase the probability amplitude of the right answer relative to all the others, making the computation likely to succeed.

Which student(s), if any, do you agree with? Explain.

83. Consider the following conversation between two students:

**Student 1:** If an  $N$ -qubit quantum computer doesn't run many instances of the same classical algorithm at once, how is it faster than an  $N$ -bit classical computer?

**Student 2:** To keep using Grover’s algorithm as an example, it would take you on average  $\frac{M}{2}$  classical searches to find the right answer among  $M$  possibilities. But the quantum Grover’s algorithm can get you the right answer in roughly  $\sqrt{M}$  searches. This “quantum advantage” becomes more and more apparent the bigger  $M$  gets.

**Student 1:** What do you mean by “quantum advantage”?

**Student 2:** It takes a certain amount of time to run all the operations that you need to get the answer that you want. If, to solve a particular problem, a quantum computer can run all of its quantum operations faster than a classical computer can run its classical operations, then there is said to be a “quantum advantage” for that problem.

Which student(s), if any, do you agree with? Explain.

84. Consider the following conversation between two students:

**Student 1:** In a particular basis, such as the  $\{|0\rangle, |1\rangle\}$  basis, a single qubit can be in one basis state,  $|0\rangle$ , or the other,  $|1\rangle$ . It can also be in a superposition state,  $a|0\rangle + b|1\rangle$ .

**Student 2:** I agree. And when you have multiple qubits, your basis will have more states. For example, the standard basis for a two-qubit system is  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ , and once again the system can be in one of the four basis states, or a superposition of up to all four.

**Student 1:** So superposition is possible for both a single qubit and a system of many qubits together. And when multiple qubits are in a superposition state, they are said to be entangled.

**Student 2:** I agree that superposition is possible for one qubit and multiple qubits, but while superposition is a necessary criterion for entanglement, it does not guarantee entanglement. It’s definitely possible for multiple qubits to be in a superposition state without being entangled.

**Student 1:** What do you mean?

**Student 2:** An operational definition for entanglement is that two qubits are entangled if there is some correlation between their measurement outcomes.

**Student 1:** Is it possible to tell if the qubits are entangled/interacting, or non-entangled/non-interacting, by inspecting the state?

**Student 2:** Well, if you can factor the state of the multi-qubit system into individual qubit states, that would be equivalent to saying that they are independent. For example,  $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) = \frac{1}{2}(|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)$  would factor into  $\frac{1}{\sqrt{2}}(|0\rangle +$

$|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ . Then try factoring  $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  into two separate qubit states—you can't do it.

**Student 1:** So any multi-qubit state that can't be factored like that is entangled?

**Student 2:** Exactly.

**Student 1:** Of course, a multi-qubit system that is in one of the basis states, and therefore isn't in a superposition state, e.g.,  $|000 \dots 0\rangle$ , can't possibly be in an entangled state.

**Student 2:** I agree with that. An entangled state is obtained only when different qubits are allowed to interact with each other physically by applying a quantum gate, such as a control not (CNOT) gate.

Which student(s), if any, do you agree with? Explain.

<i>Checkpoints</i>		
<ul style="list-style-type: none"> <li>• Though an <math>N</math>-qubit quantum computer can be in a superposition of <math>2^N</math> basis states during a computation, the coefficients of each of those <math>2^N</math> basis states need not be kept track of on classical registers. This information remains inside the computer and cannot be obtained through a single measurement.</li> <li>• Quantum algorithms are not at all similar to running many instances of classical algorithms in parallel; they take advantage of superposition, entanglement, and quantum measurement, none of which have classical analogues.</li> <li>• Quantum computation encodes all possible answers within mutually exclusive basis states that form a superposition. If one tries to read the output by making a measurement in the initial state or in an intermediate superposition state, the correct answer is not likely to be obtained. Good quantum algorithms manipulate the superposition to make amplitudes of the wrong answers cancel and amplitudes of the right answers add up, increasing the chances of obtaining a correct answer when the output is measured at the end of a computation.</li> </ul>		
<b>System</b>	<b>Can it be in a superposition state?</b>	<b>Can it be in an entangled state?</b>
1 qubit	Yes (but this depends on the choice of basis)	No
Multiple qubits	Yes	Only for some superposition states (ones that cannot be factored into individual qubit states)  Physical interaction between qubits is necessary for the

		states of different qubits to be entangled.
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### **Mini-section 3: Measuring the output**

*You should be able to:*

- Describe the output of a quantum computer after a computation is finished
- Discuss the nature of the information contained in the output

*Orientation*

#### **What is the output of a quantum computer like?**

Quantum computers are not magic. While it may seem that they can take advantage of vastly more information than classical computers of the same size, any output of a quantum computer with some number of qubits does not in fact impart more information than could be obtained from a classical computer with the same number of bits.

A quantum computer does not become a classical computer once its output is measured (and thus “read”). However, any properties specific to quantum mechanics that it may have had, including superposition and entanglement, are eliminated through collapse of the quantum state in the measurement basis, and we obtain  $N$  classical bits of information.

85. Consider the following conversation between several students:

**Student 1:** When an  $N$ -qubit quantum computer is finished with its computation, we “read” an output, which is equivalent to measuring each qubit. In general, the quantum state of an  $N$ -qubit system can be in a superposition of  $2^N$  basis states in the computation, but the measurement collapses the state of the quantum computer into just one of the  $2^N$  basis states.

**Student 2:** That output is a string of  $2^N$  classical bits, because of the  $2^N$  total coefficients in the superposition, right?

**Student 3:** No, the output must be  $N$  classical bits, because you get one bit of classical information from measuring each qubit.

**Student 4:** I agree with Student 3. You can’t actually know what the values of the coefficients in the superposition are from just a single measurement. Since your measured output will only ever give you  $N$  classical bits of information, you can never find out from the quantum computer how likely it was for you to get that answer by just running a quantum computer once. It’s like if you flipped a coin and got heads, you don’t know from that single result how likely it was to get heads.

Which student(s), if any, do you agree with? Explain.

86. Consider the following conversation between several students:

**Student 1:** When a quantum computer is finished evolving its qubits with quantum gates, we simply make a single measurement in its final state to get the correct answer with 100% certainty.

**Student 2:** No, there is a chance that we could get the wrong answer, because the final state might still have a probability spread instead of being peaked like a delta function at the right answer.

**Student 3:** Actually, in cases where the answer is easy to check, all we have to do is run the quantum program several times starting from the same initial state, and make measurements until we get an output that solves the problem. For example, if you want to factorize the product of two large prime numbers using a quantum computer, you can just multiply the output numbers and see if you get the original number. If not, repeat the computation starting from the same initial state.

**Student 1:** But when we determined a time to make our measurement on the quantum computer, wasn't the whole point to read the output when we will obtain the correct answer with probability 1 and all other answers with probability 0? Why isn't the final state a delta function peaked at the correct answer?

**Student 3:** We definitely want to have a reasonable chance of measuring the correct answer at the end of computation, but the peaked delta function is an idealization. In practice for most problems, measurements will not give the correct answer with 100% certainty even if the qubits were not affected by decoherence. For our purposes, a high probability of obtaining the correct answer is still enough to outperform a classical computer, which could take longer than the age of the universe to solve certain problems like factoring products of very large prime numbers.

Which student(s), if any, do you agree with? Explain.

87. Quantum algorithms do not always give the correct answer with 100% probability. Suppose you have a quantum algorithm for a three-qubit quantum computer, and the correct answer (verified through some classical methods) is 101 (mapped to the state  $|101\rangle$ ). For each of the following states, what is the probability that the quantum computer will give the correct answer if measurement is made in the standard basis? What is the probability that the quantum computer will give the wrong answer?

(A)  $\frac{\sqrt{7}}{\sqrt{8}}|101\rangle + \frac{i}{\sqrt{8}}|111\rangle$

(B)  $\frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{2}}|101\rangle$

$$(C) \frac{1}{\sqrt{8}}|000\rangle + \frac{i}{\sqrt{8}}|001\rangle + \frac{1}{\sqrt{8}}|010\rangle + \frac{1}{\sqrt{8}}|011\rangle + \frac{i}{\sqrt{8}}|100\rangle + \frac{1}{\sqrt{8}}|101\rangle + \frac{i}{\sqrt{8}}|110\rangle + \frac{1}{\sqrt{8}}|111\rangle$$

$$(D) \frac{1}{10}|000\rangle + \frac{1}{10}|001\rangle + \frac{1}{10}|010\rangle + \frac{1}{10}|011\rangle + \frac{1}{10}|100\rangle + \sqrt{\frac{93}{100}}|101\rangle + \frac{1}{10}|110\rangle + \frac{1}{10}|111\rangle$$

$$(E) \sqrt{\frac{1}{7000}}|000\rangle + \sqrt{\frac{1}{7000}}|001\rangle + \sqrt{\frac{1}{7000}}|010\rangle + \sqrt{\frac{1}{7000}}|011\rangle + \sqrt{\frac{1}{7000}}|100\rangle + \sqrt{\frac{999}{1000}}|101\rangle + \sqrt{\frac{1}{7000}}|110\rangle + \sqrt{\frac{1}{7000}}|111\rangle$$

88. Consider the following conversation between several students:

**Student 1:** As for the output of a quantum computer, I disagree with talking about it in terms of classical bits. You can only have quantum inputs and outputs, not classical ones.

**Student 2:** It's true that the physical qubits don't magically turn into classical bits. It's more appropriate to say that, at the end of a computation, when you make a measurement in any basis to read the output, you get an amount of information equivalent to  $N$  classical bits. The superposition of  $N$  qubit states, which would include up to  $2^N$  basis states, would collapse into a single one of those basis states yielding  $N$  bits of classical information.

**Student 3:** For example, if  $\{|0\rangle, |1\rangle\}$  is the measurement basis for each qubit, you will only ever read out 0 or 1, giving you 1 bit of information in either case. This is true even if the state right before the measurement was  $a|0\rangle + b|1\rangle$ . Similarly, for a 2-qubit quantum computer, you would get 2 bits of information, corresponding to a collapse to  $|00\rangle, |01\rangle, |10\rangle, \text{ or } |11\rangle$ , even if the state before the measurement was  $a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ .

**Student 2:** I agree with Student 3. In fact, for each added qubit, the number of basis states doubles. This means the number of basis states grows exponentially as  $2^N$  with increasing  $N$ .

Which student(s), if any, do you agree with? Explain.

### Checkpoints

- The output of a quantum computer is a single state containing  $N$  classical bits of information (an  $N$ -bit string).
- A system of  $N$  qubits does not give access to more information than a system of  $N$  classical bits. Although it is possible for a quantum computer to explore all possible combinations of states of the  $N$  qubits during the computation, only one of these combinations can be read out at the end. This is the step during which the system is “measured” and collapses to a single measurement basis state, giving us one bit of classical information for each qubit.

### Summary

Classical computers process information encoded in bits. By analogy, quantum computers process quantum information encoded in qubits. The more qubits there are in the quantum computer, the more quantum information can be encoded.

When one runs a computation on an  $N$ -qubit quantum computer, one can produce the same number of outputs that an  $N$ -bit classical computer can produce. However, quantum computers take advantage of a completely different type of calculation method, using quantum states and involving superposition and entanglement. Designing clever algorithms can allow them to perform certain computations more efficiently than classical computers. However, the advantages are only apparent when the system size gets large enough.

Because measurements in quantum mechanics are probabilistic rather than deterministic, and the answer to a quantum computation is obtained through measurement, it is typically still possible for the computer to output an incorrect answer. Thus, for a quantum computer to be effective at solving a problem, it must be difficult for a classical computer to calculate the solution, yet easy to check the correctness of the quantum computer’s outputs (solutions). Even running the same quantum computation several times would still be much, much faster than attempting to do the same computation classically (such a case is known as *quantum advantage*), so it is not critically detrimental if the output of a quantum computer for a given problem is occasionally inaccurate.

## Appendix F.2 Pre-test and post-test questions

The questions that students were given for the post-test are reproduced here. All parts of Questions 1-8 were given to students in the QCQI class and both Physics classes. All parts of Questions 9-11 were given to students in the QCQI class and only one of the Physics classes (with the exception of 10d, which was given to only the Physics class).

Students were given the following information:

- All states, operators, and measurements are assumed to be in the  $\{|0\rangle, |1\rangle\}$  orthonormal basis.

- A measurement made in immediate succession means that the measurement is to be made before the state has had time to evolve.
- The symbol  $\doteq$  means “is represented by” in a given basis.

1. This question is about multiple bits and qubits.

(A) How many total states are possible for a system of 2 classical bits? Write them down (e.g., “00” can represent the state in which both bits take values of 0).

(B) How many total linearly independent states are possible for a system of 2 quantum bits (qubits)? Write down a representation of these states using the notation of kets, e.g.,  $|00\rangle$ , ...

(C) Give an example of a physical system that can be used as a classical bit, and one that can be used as a qubit.

(D) The quantum state of the system of two qubits can be represented as  $a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ . If we make a measurement on this system, what are the probabilities of obtaining each of the basis states  $|00\rangle$ ,  $|01\rangle$ ,  $|10\rangle$ , and  $|11\rangle$ ? What should you obtain by adding these 4 probabilities together? (Note: In general, assume complex numbers  $\{a_{ij}\}$  where  $i, j = 0, 1$ .)

2. Consider the state  $|q\rangle = \sqrt{\frac{11}{13}}|0\rangle + \sqrt{\frac{2}{13}}|1\rangle$ .

(A) If we perform a measurement on a system prepared in the above state  $|q\rangle$ , what is the probability of obtaining the state  $|0\rangle$ ? What is the probability of obtaining the state  $|1\rangle$ ?

(B) How might you construct a gate such that, after its application to the state  $|q\rangle$ , a measurement will yield the state  $|0\rangle$ ?

3. Consider the state  $|q\rangle = |1\rangle$ . In the  $S_z$  basis,  $|1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(Note: In terms of the  $|0\rangle$  and  $|1\rangle$  states, the states  $|+\rangle$  and  $|-\rangle$  are defined as  $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ .)

(A) If a qubit in the state  $|q\rangle$  is measured in the  $\{|+\rangle, |-\rangle\}$  basis, what is the probability of obtaining the state  $|-\rangle$ ?

(B) Suppose the measurement in part (A) yielded the state  $|-\rangle$ . If we make a successive measurement in the  $\{|0\rangle, |1\rangle\}$  basis, what are the possible outcomes, and the probabilities of those outcomes?

- (C) Suppose we first act upon the state  $|q\rangle = |1\rangle$  with the Hadamard gate,  
 $\hat{H} \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ . What is the state after the gate has been applied,  $\hat{H}|q\rangle$ ? Is it possible for a measurement in the state  $\hat{H}|q\rangle$  to yield  $|0\rangle$ , and if so, with what probability?
- (D) Can you construct a quantum gate  $\hat{G}$  such that a measurement on the state  $\hat{G}|q\rangle$  will then yield the state  $|0\rangle$  with 100% certainty?
4. Indicate whether the following statements are true or false, and explain your reasoning in each case:
- (A) To initialize an  $N$ -bit classical computer, one must set  $N$  bit values to zero. To initialize an  $N$ -qubit quantum computer, one must set  $2^N$  qubit states to  $|0\rangle$ .
- (B) When an  $N$ -qubit quantum computer completes a computation and we “read” the output in the  $\{|0\rangle, |1\rangle\}$  basis for each qubit, we obtain information equivalent to that contained in  $N$  classical bits.
- (C) Large quantum computers, e.g., with 1000 qubits, cannot be made because they cannot possibly process  $2^{1000} \cong 10^{300}$  variables, which is more than the number of atoms in the known universe.
- (D) If the quantum state of two qubits is in a superposition state in the basis consisting of  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ , then the state is said to be entangled.
5. Choose one of the following two statements and explain why you agree with it:
- Statement 1:** In an  $N$ -bit classical computer, the computer has  $N$  available states during the calculation, but in an  $N$ -qubit quantum computer, there are  $2^N$  linearly independent states.
- Statement 2:** In an  $N$ -bit classical computer, the computer has  $2^N$  available states during the calculation, and in an  $N$ -qubit quantum computer, there are  $2^N$  linearly independent states.
6. How many operations must be performed to initialize a quantum computer with  $N$  qubits?
- $N$
  - $2N$
  - $2^N$
  - None of the above

7. Quantum algorithms do not always give the correct answer with 100% probability. Suppose you have a quantum algorithm for a two-qubit quantum computer in which the final state is  $|q\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ , and the correct answer (verified through some classical methods) corresponds to 01 (the corresponding state is  $|01\rangle$ ).

(A) Suppose the initial state of the quantum computer is  $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ , i.e.  $a_{ij} = \frac{1}{2}$ ,  $i, j = 0, 1$ . What is the probability that the quantum computer will give the correct answer? What is the probability that the quantum computer will give the wrong answer?

(B) What must  $\{a_{ij}\}$  be for a measurement to yield the correct answer  $|01\rangle$  100% of the time? In a quantum computer, how can  $\{a_{ij}\}$  be made to take these values if the initial state is the one in part (A)?

(C) If the state of the quantum computer is made to be  $\sqrt{\frac{1}{300}}|00\rangle + \sqrt{\frac{99}{100}}|01\rangle + \sqrt{\frac{1}{300}}|10\rangle + \sqrt{\frac{1}{300}}|11\rangle$ , what is the probability that the quantum computer will give the correct answer? What is the probability that the quantum computer will give the wrong answer?

8. Answer the following questions:

(A) Can a single qubit be in a superposition state? Explain.

(B) Can a single qubit state be an entangled state? Explain.

(C) Can a multi-qubit state be an entangled state? Explain.

(D) Can a multi-qubit state be an entangled state without any prior interactions between the qubits (either directly between two qubits or via other qubits)? Explain.

*\*The questions below were not asked in physics class 1.*

9. Consider the state  $|q\rangle = \sqrt{\frac{11}{13}}|0\rangle + \sqrt{\frac{2}{13}}|1\rangle$ .

(A) If the result of a first measurement made in this state was  $|0\rangle$ , and another measurement is made in immediate succession before the state has had time to evolve, what is the probability of obtaining the state  $|0\rangle$ ? What is the probability of obtaining the state  $|1\rangle$ ?

(B) If instead the result of the first measurement was  $|1\rangle$ , and another measurement is made in immediate succession, what is the probability of obtaining the state  $|0\rangle$ ? What is the probability of obtaining the state  $|1\rangle$ ?

- (C) If we perform a measurement on a large number of systems, each prepared in the above state  $|q\rangle$ , what fraction of those measurements will yield the state  $|0\rangle$ ? What fraction will yield the state  $|1\rangle$ ?

10. Consider the state  $|q\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ .

- (A) What is the corresponding bra state for this ket state?
- (B) A general operator in a two-dimensional vector space can be written in the form  $a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$ . Find the values of  $a, b, c$ , and  $d$  that the identity operator  $\hat{I}$  takes in this 2-D vector space.
- (C) A general operator can be written in the form  $a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$ . Find the values of  $a, b, c$ , and  $d$  that the outer product  $|q\rangle\langle q|$  takes.
- (D) Calculate  $|q\rangle\langle q|1\rangle$ . (This is the action of the operator  $|q\rangle\langle q|$  on the state  $|1\rangle$ .)  
*\*This question was not asked in the QCQI class.*

11. Write an expression for the Hadamard gate using Dirac notation.

### Appendix F.3 Additional data

Following are tables that break down the data in other potentially interesting ways. Tables 3-4 display results from both Physics classes separately, and we were able to collect retention data for the second Physics class, which are shown in Table 4. Table 5 presents results from the physics majors and non-physics majors in the QCQI course separately.

**Appendix Table 1 Physics class 1,  $N = 13$ .**

Question	Pre	Post	Norm. Gain	Effect size
1a	100%	100%	--	--
1b	96%	100%	1.00	0.39
1c	62%	88%	0.70	0.81
1d	71%	88%	0.60	0.88
2a	100%	96%	--	--
2b	38%	50%	0.19	0.35
3a	92%	92%	0.00	0.00
3b	85%	100%	1.00	0.58
3c	88%	96%	0.67	0.33
3d	50%	54%	0.08	0.11
4a	12%	69%	0.65	1.68
4b	38%	92%	0.88	1.67
4c	46%	85%	0.71	1.21
4d	23%	65%	0.55	1.04
5	23%	81%	0.75	1.40
6	8%	62%	0.58	1.32
7a	77%	92%	0.67	0.42
7b	38%	73%	0.56	0.86
7c	85%	100%	1.00	0.58
8a	62%	92%	0.80	1.07
8b	62%	96%	0.90	1.01
8c	85%	96%	0.75	0.59
8d	54%	85%	0.67	0.76

**Appendix Table 2 Physics class 2,  $N = 22$ . We were able to collect retention data for some of these questions, which were asked on the final exam several months after the post-test. Gain and effect size are measured from the pre-test.**

Question	Pre	Post	Norm. Gain	Effect size	Final	Norm. Gain	Effect size
1a	98%	100%	1.00	0.31			
1b	100%	100%	-	-			
1c	64%	93%	0.81	0.95			
1d	94%	94%	0.00	0.00			
2a	100%	100%	--	--			
2b	30%	50%	0.29	0.74			
3a	95%	100%	1.00	0.31			
3b	86%	98%	0.83	0.49			
3c	64%	89%	0.69	0.67			
3d	43%	77%	0.60	1.06			
4a	41%	89%	0.81	1.22	98%	0.96	1.67
4b	64%	100%	1.00	1.19	98%	0.93	1.08
4c	57%	91%	0.79	1.14	100%	1.00	1.76
4d	75%	95%	0.82	0.62	95%	0.80	0.60
5	55%	91%	0.80	0.93	73%	0.57	0.79
6	36%	91%	0.86	1.38			
7a	82%	100%	1.00	0.80			
7b	61%	80%	0.47	0.48			
7c	89%	100%	1.00	0.54			
8a	75%	91%	0.64	0.44			
8b	77%	95%	0.80	0.58			
8c	82%	98%	0.88	0.61			
8d	66%	89%	0.67	0.61			
9a	82%	100%	1.00	0.72			
9b	82%	100%	1.00	0.72			
9c	91%	95%	0.50	0.20			
10a	91%	95%	0.50	0.27			
10b	70%	77%	0.23	0.21	85%	0.49	0.43
10c	66%	89%	0.67	0.70			
10d	57%	89%	0.74	0.82			
11	30%	73%	0.61	1.03			

**Appendix Table 3** The data presented in the QCQI course in Table 1 ( $N = 18$ ), with only the physics majors shown ( $N = 11$ ). Pre-test and post-test average scores, normalized gain, and effect size as measured by Cohen's  $d$  are presented. Student data for the pre- and post-test in each case are matched. An additional column is included containing (unmatched) data from all students who completed the post-test ( $N = 12$ ).

Question	Pre	Post	Norm. Gain	Effect size	Post (all)
1a	100%	91%	-	-	92%
1b	100%	95%	-	-	96%
1c	59%	82%	0.56	0.64	79%
1d	82%	93%	0.63	0.63	94%
2a	100%	100%	-	-	100%
2b	59%	77%	0.44	0.56	75%
3a	100%	91%	-	-	83%
3b	95%	100%	1.00	0.43	100%
3c	77%	91%	0.60	0.48	92%
3d	50%	68%	0.36	0.56	63%
4a	50%	86%	0.73	0.86	88%
4b	73%	86%	0.50	0.37	88%
4c	50%	95%	0.91	1.36	96%
4d	64%	86%	0.63	0.54	88%
5	45%	77%	0.58	0.68	79%
6	45%	82%	0.67	0.78	83%
7a	73%	100%	1.00	0.83	100%
7b	50%	82%	0.64	0.80	79%
7c	77%	100%	1.00	0.78	100%
8a	91%	86%	-	-	88%
8b	95%	100%	1.00	0.43	100%
8c	86%	93%	0.50	0.34	94%
8d	70%	91%	0.70	0.57	92%
9a	82%	100%	1.00	0.64	100%
9b	82%	100%	1.00	0.64	100%
9c	95%	100%	1.00	0.43	100%
10a	82%	100%	1.00	0.76	100%
10b	64%	68%	0.13	0.12	71%
10c	59%	80%	0.50	0.58	79%
11	27%	68%	0.56	0.94	63%

**Appendix Table 4** The data presented in the QCQI course in Table 1 ( $N = 18$ ), with only the non-physics majors shown ( $N = 7$ ). Pre-test and post-test average scores, normalized gain, and effect size as measured by Cohen's  $d$  are presented. Student data for the pre- and post-test in each case are matched. An additional column is included containing (unmatched) data from all students who completed the post-test ( $N = 16$ ).

Question	Pre	Post	Norm. Gain	Effect size	Post (all)
1a	100%	100%	-	-	100%
1b	100%	89%	-	-	95%
1c	57%	100%	1.00	1.35	84%
1d	93%	93%	-	-	92%
2a	100%	100%	-	-	100%
2b	43%	50%	0.13	0.19	41%
3a	100%	100%	-	-	100%
3b	71%	100%	1.00	0.83	94%
3c	79%	100%	1.00	0.77	94%
3d	43%	71%	0.50	0.93	66%
4a	50%	86%	0.71	1.06	91%
4b	43%	93%	0.88	1.45	91%
4c	64%	86%	0.60	0.67	80%
4d	71%	100%	1.00	1.51	84%
5	57%	86%	0.67	0.62	91%
6	29%	86%	0.80	1.31	81%
7a	86%	86%	-	-	91%
7b	43%	71%	0.50	0.68	72%
7c	57%	100%	1.00	1.13	100%
8a	79%	100%	1.00	0.77	100%
8b	79%	100%	1.00	0.77	97%
8c	71%	86%	0.50	0.44	88%
8d	25%	86%	0.81	1.52	75%
9a	100%	100%	-	-	94%
9b	100%	100%	-	-	100%
9c	86%	86%	-	-	94%
10a	57%	100%	1.00	1.76	97%
10b	57%	64%	0.17	0.15	72%
10c	29%	64%	0.50	0.93	58%
11	21%	57%	0.45	0.76	34%

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