Chapter 18

Seeing in Special and General Relativity

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We compute how objects appear at relativistic speeds and in the vicinity of a large spherically symmetric mass.

18.1 Special Relativity

How do objects appear at relativistic speeds? The Lorentz-Fitzgerald length contraction in the direction of motion is not the only effect that needs to be considered when determining the apparent shape of an object. A single observer forms an image of an object by collecting light emitted from the entire object. When an observer sees the object, she does not see its current position nor its true shape, but sees each part of the object where it was when the light was emitted. This position is known as the retarded position. Because of the finite speed of light, we must calculate when and where along the object’s trajectory each light ray originated to determine the image formed on the observer’s retina.

The relative velocity of an object with respect to a single observer defines a direction, which we take to be the direction of the $x$-axis in the observer’s frame of reference, $S$. Let $S'$ be the rest frame of the object and $v$ be the velocity of $S$ with respect to $S'$, such that the origins coincide at $t = t' = 0$. The Lorentz transformation connecting $S$ and $S'$ is

\begin{align}
  x' &= \frac{1}{\gamma}(x - vt) \quad (18.1a) \\
  y' &= y \quad (18.1b) \\
  z' &= z \quad (18.1c) \\
  t' &= \gamma(t - vx/c^2) \quad (18.1d)
\end{align}
where $\gamma = 1/\sqrt{1 - v^2/c^2} = 1/\sqrt{1 - \beta^2}$ and $\beta = v/c$. In the rest frame of the object, the spatial separation between two points on the object is

$$d' = \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2}.$$ (18.2)

In the rest frame of the observer, the separation is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$ (18.3)

where

$$x_2 - x_1 = \gamma(x'_2 - x'_1)$$ (18.4a)
$$y_2 - y_1 = y'_2 - y'_1$$ (18.4b)
$$z_2 - z_1 = z'_2 - z'_1.$$ (18.4c)

The change in the $x$ separation is known as the Lorentz-Fitzgerald contraction. If we know the shape of the object in the rest frame, we can determine the shape in the observer’s frame by applying an affine transformation (see Chapter 17) that rescales the object’s $x$ dimension by $\gamma$. Listing 18.1 shows how this transformation is done using a two-dimensional wire frame model of a ring. The `ContractedRing` class defines a ring of unit radius in the object’s rest frame using an array of `Point2D` objects. `Point2D` represents a location and can be transformed because it is part of the standard Java 2D API. These points are transformed into the observer’s frame and the transformed shape is drawn by connecting the points using line segments.

Listing 18.1: The `ContractedRing` class implements the Lorentz-Fitzgerald contraction of a ring moving in the $x$ direction.

```java
package org.opensourcephysics.sip.ch18;
import java.awt.*;
import java.awt.geom.*;
import org.opensourcephysics.display.*;

public class ContractedRing implements Drawable {
    double vx = 0, time = 0;
    Point2D[] labPoints, pixPoints;

    public ContractedRing(double x0, double y0, double vx, int numberOfPoints) {
        labPoints = new Point2D[numberOfPoints];
        pixPoints = new Point2D[numberOfPoints];
        double theta = 0, dtheta = 2*Math.PI/(numberOfPoints - 1);
        // unit radius circle
        for(int i = 0; i < numberOfPoints; i++) {
            double x = Math.cos(theta); // x coordinate
            double y = Math.sin(theta); // y coordinate
            labPoints[i] = new Point2D.Double(x, y);
            theta += dtheta;
        }
        this.vx = vx;
    }
}
```
Exercise 18.1. Lorentz-Fitzgerald contraction

Write a test program that instantiates and displays a ContractedRing object. Measure the dimensions of the on-screen object to verify that the Lorentz-Fitzgerald contraction is computed correctly.

We now introduce retardation effects. Let $r = (x, y)$ be the current location of an arbitrary point on the object as shown in Figure 18.1. Because of the finite speed of light, an observer at the origin sees a point moving with a speed $v$ in the $x$ direction not at its current location, but at the previous location

$$r_{\text{ret}} = (x - \delta, y).$$

(18.5)

The $x$ coordinate is retarded by $\delta = v\tau$, where $\tau$ is the travel time of light from $r_{\text{ret}}$ to the observer. The distance from the retarded position to the observer is

$$r_{\text{ret}} = \sqrt{(x - \delta)^2 + y^2}.$$  (18.6)
Figure 18.1: The geometry used to derive the retarded position and time for an observer at the origin. The observed point is moving with constant speed \( v \) in the \( x \) direction.

We use the speed of light to convert distance to light travel time (\( r_{\text{ret}} = c\tau \)), substitute for \( \delta \), and obtain

\[
ct = \sqrt{(x - vt)^2 + y^2}. \tag{18.7}
\]

If we square both sides and solve for \( \tau \), we find

\[
\tau = \frac{-x\beta + \sqrt{x^2\beta^2 + (1 - \beta^2)(x^2 + y^2)}}{c(1 - \beta^2)}, \tag{18.8}
\]

where we have chosen the positive square root to make the travel time \( \tau \) positive.

We can incorporate the time delay (18.8) into a program to obtain the image seen by a stationary observer. We subclass \texttt{ContractedRing} and add methods to compute and draw the retarded positions of the points on the ring. Retarded points are stored in an array and the retarded positions are computed by solving (18.8). The class \texttt{ObservedRing} is shown in Listing 18.2.

Listing 18.2: The \texttt{ObservedRing} class models the appearance of a ring traveling in the \( x \) direction at relativistic speeds.
Exercise 18.2. Relativistic ring
Write a test program that instantiates and displays the apparent shape of a rapidly moving ring. Explain the sharp convex point when the front edge of the ring touches (reaches) the observer. Does the ring ever appear to be concave? Why?

Exercise 18.3. Relativistic ruler
Modify the ObservedRing class to display a long narrow rectangle. Can an observer see the Lorentz-Fitzgerald contraction if this “ruler” is moving along the x-axis? Click-drag within the display to measure the apparent length of the ruler at various positions. Explain the meaning of the term observer in relativity. Some authors (see Taylor and Wheeler) prefer the term bookkeeper. Why might this term be better?

What an observer “sees” is quite different from what is given by the Lorentz contraction. What makes Einstein’s special theory of relativity profound is not the appearance. It is that length really does contract and time really does slow down.

Exercise 18.4. Relativistic square
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Write a target class that instantiates and displays the apparent shape of a moving square whose trajectory is \((x_0 - vt, b)\) past a stationary observer at \((0, 0)\). Is the apparent shape still a square? Explain why the observer can see the square’s hidden side. This effect is known as \textit{Terrell rotation}.

We can rotate the shape seen in the simulation around the \(x\)-axis to visualize the apparent shape of a three-dimensional object approaching an observer head-on. The case for a sphere was treated analytically by Suffern, but most other two- and three-dimensional objects cannot be treated analytically and are best visualized using the help of a computer. A complete and accurate visualization must take into account additional physics, such as the Doppler effect and angular changes in the intensity distribution of the emitted light due to acceleration (see Weisskopf).

18.2 General Relativity

The idea that space is curved was first tested by Gauss who measured the interior angles of a large triangle by placing lanterns on three mountain tops. Although Gauss obtained the Euclidian (flat-space) result of 180°, measurements of stellar positions during the 1919 total solar eclipse by Eddington showed that the sum of the interior angles is not 180°. It is an experimental fact that the universe is non-Euclidian.

The Eddington experiment was remarkable because it confirmed Einstein’s general theory of relativity and showed that space and time are not separate entities. We cannot measure space, only distances between events in space using rulers, light beams, and clocks. Furthermore, the separation between events is not the same for different observers unless they incorporate both spatial and temporal separations into their definition of distance. In the absence of gravitational fields, observers moving at constant relative velocity can reconcile (18.2) and (18.3) and obtain the same “distance” only if they agree that the distance between events, \(\Delta \sigma\), includes time and is measured as

\[
\left(\Delta \sigma\right)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2,
\]

where \(c\) is the speed of light, \(\Delta t\) is the temporal separation, and \(\Delta x, \Delta y, \) and \(\Delta z\) are the spatial coordinate separations. Equation (18.9) is based on Einstein’s special theory of relativity and is known as the Minkowski metric. It follows from Einstein’s assumption that Maxwell’s equations must be the same for all observers in uniform relative motion and leads naturally to the equivalence of mass and energy embodied in the famous equation \(E = mc^2\).

Einstein’s great insight that acceleration and gravity are indistinguishable enabled him to incorporate gravity into the spacetime fabric by generalizing (18.9). Imagine an elevator compartment resting on the surface of Earth in which the occupants perform experiments, such as dropping an object or observing a swinging pendulum, that reveal the presence of Earth’s gravity. Then the occupants are placed in a compartment far from any gravitational object, and the compartment accelerates at 9.8 m/s². According to Einstein, the experimental results must be identical. Furthermore, if the near-Earth elevator cable is cut to produce a freely falling reference frame, then the occupants will be unable to detect the gravitational field. The implication is that we can do away with gravity and regard it as a consequence of an accelerated reference frame in four-dimensional spacetime. It took Einstein ten years to incorporate this equivalence of gravitational forces and accelerated motion to the special theory of relativity to produce the general theory.
Einstein’s general theory of relativity produces ten simultaneous coupled non-linear partial differential equations. Calculations using this theory are truly daunting and require sophisticated mathematical techniques such as tensor analysis and Riemannian geometry. All forms of energy are subject to the effects of gravity and nonlinearities arise because a body’s gravitational field is itself a form of energy and therefore is also subject to the effects of gravity. Few analytical results are known. Two of the most important are the Schwarzschild and Kerr metrics in the vicinity of a spherically symmetric mass. Except for very special cases or very weak fields, the dynamical equations using these metrics must be solved numerically to predict how particles move and how they appear when seen by an observer.

18.3 Dynamics in Polar Coordinates

General relativistic trajectories of particles and light in the vicinity of spherically symmetric gravitational fields are conveniently described using polar coordinates. We therefore reformulate the classical two-body problem (see Chapter 5) using polar coordinates. If the motion is confined to a plane, rectangular coordinates, \((x, y)\), and polar coordinates, \((r, \phi)\), are related by

\[
x = r \cos \phi, \quad y = r \sin \phi
\]

and

\[
r = \sqrt{x^2 + y^2}, \quad \phi = \arctan \frac{y}{x}.
\]

The radial velocity is given by

\[
\dot{r} = \frac{dr}{dt} = \frac{r \cdot \mathbf{v}}{r},
\]
and the angular velocity is given by
\[ \dot{\phi} = \frac{d\phi}{dt} = \frac{L}{mr^2}, \] (18.13)
where \( L \) is the magnitude of the conserved angular momentum \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \).

To construct the appropriate differential equations, the radial and angular accelerations can be obtained by differentiating (18.12) and (18.13) with respect to time. Another approach is to use the Lagrangian
\[ \mathcal{L} = \frac{1}{2} \left[ r^2 \dot{r}^2 + r^2 \dot{\phi}^2 \right] + \frac{GM}{r}, \] (18.14)
and apply Lagrange’s equations of motion:
\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = 0, \quad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \] (18.15)

If we do the differentiation, we obtain the following rate equations for the polar state vector \((r, \dot{r}, \phi, \dot{\phi}, t)\).

\[ \frac{dr}{dt} = \dot{r} \] (18.16a)
\[ \frac{d\dot{r}}{dt} = r \ddot{\phi}^2 - \frac{GM}{r^2} \] (18.16b)
\[ \frac{d\phi}{dt} = \dot{\phi} \] (18.16c)
\[ \frac{d\dot{\phi}}{dt} = -\frac{2}{r} \dot{\phi} \dot{r} \] (18.16d)
\[ \frac{dt}{dt} = 1. \] (18.16e)

**Exercise 18.5.** Angular momentum
Show that (18.16) leads to the polar coordinate expression for conservation of angular momentum
\[ r \ddot{\phi} + 2 \dot{r} \dot{\phi} = 0. \] (18.17)

**Exercise 18.6.** Classical trajectories
Modify the PlanetApp program introduced in Chapter 5 so that the classical trajectory of a particle is calculated using polar variables rather than cartesian variables. Use (18.10)–(18.13) to determine the initial state and compare your results with those of the PlanetApp program.

The Open Source Physics plotting panel contains an axis object that displays a cartesian coordinate grid by default. This grid can be replaced by a polar by using the method setPolar (see Figure 18.2):

```
plotFrame.setPolar("Trajectory", 1.0);
```

The first parameter is the plot’s title and the second parameter is the radial grid separation. The new plotting panel also displays the polar coordinates in the bottom left when the mouse is clicked or dragged.
Exercise 18.7. Polar coordinates
Modify Exercise 18.6 so that polar coordinate values are displayed when the mouse is click-dragged within the display.

18.4 Black Holes and Schwarzschild Coordinates

Our goal is to compute the worldlines (trajectories in spacetime) of particles and light in the vicinity of spherically symmetric gravitational objects. Readers should consult the classic text, Exploring Black Holes by Edwin Taylor and John Wheeler, for a more complete discussion of the physics near objects that have undergone gravitational collapse. Such an object is known as a black hole because light cannot escape from its vicinity. We will calculate the general relativistic trajectories of particles and light near a spherically symmetric gravitational mass. Because physical space is non-Euclidian, a two-dimensional plot of these trajectories will be distorted. Unlike classical orbits, the general relativistic orbits appear very different when seen by a viewer in the real world. We must calculate the trajectories of multiple light rays to construct the view as seen by a single observer.

Because time is incorporated as a fourth dimension and because space is curved, a general relativistic coordinate system centered on a spherically symmetric mass is more complicated than a three-dimensional Euclidean coordinate system. The azimuthal angle, $\phi$, can still be defined as the ratio of the arc length to the circumference on an imaginary circle because the spherically symmetric gravitational mass, $M$, is located at the origin. However, the radial coordinate is not defined as the physical distance from the center because sometimes a particle cannot reach the center. Rather, it is calculated using a path that circumnavigates the central mass:

$$r = \text{circumference}/(2\pi).$$

The time coordinate is defined using a wristwatch located far from the center of attraction. Note the nonlocal character of the $(r, \phi, t)$ spacetime coordinates. The wristwatch worn by the surveyor circumnavigating the mass used to measure $r$ is not the time used to record events at that value of $r$.

This $(r, \phi, t)$ spacetime coordinate system is known as Schwarzschild coordinates, and is a universal bookkeeping device that enables us to translate observations from one reference frame to another. The Schwarzschild coordinates give rise to a metric, known as the Schwarzschild metric, that enables us to calculate the four-dimensional distance between adjacent spacetime events. This metric is given by

$$d\sigma^2 = -d\tau^2 = -(1 - \frac{2M}{r}) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2,$$

where $t$, $r$, and $\phi$ refer to the faraway time, the radial coordinate, and the azimuthal coordinate, respectively. Because we associate distance with a positive number, it is common to use $d\sigma^2$ when the right-hand side of (18.19) is positive and to use $d\tau^2$ when the right-hand side of (18.19) is negative. As in special relativity, these two forms are referred to as the space-like form and the time-like form of the metric, respectively. Note that both time and distance have units of length in (18.19). The speed of light, $c$, is the conversion factor,

$$t_{\text{meters}} = ct_{\text{seconds}}.$$

(18.20)
Mass also has units of meters, and the conversion factor to kilograms is given in terms of the speed of light and Newton’s gravitational constant, $G$.

$$M = \frac{G}{c^2} M_{\text{kg}}. \quad (18.21)$$

If we freeze time, so that $dt = 0$, then the Schwarzschild metric predicts that two simultaneous events far from the central mass are separated by the Euclidean metric in polar coordinates,

$$d\sigma^2 = dr^2 + r^2 d\phi^2. \quad (18.22)$$

Strange things happen if two events are near the gravitational mass. The separation (known as the proper distance) between two adjacent events becomes

$$d\sigma^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \quad (18.23)$$

The proper distance $d\sigma$ is the distance measured by a surveyor placing meter sticks in space between two locations. This distance is clearly greater than the result predicted by (18.22) when the events occur at different $r$-coordinates. In fact, the rate of change of the proper length with respect to the $r$-coordinate becomes infinite as we approach what is known as the event horizon, $r = 2M$. The distance around a gravitational mass has no such singularity, which is why this distance is used to define the $r$-coordinate. (The singularity at the event horizon is an artifact of the Schwarzschild coordinate system. An object falling into a black hole passes through the event horizon without incident, and is crushed only at $r = 0$.)

**Exercise 18.8.** Measuring distance

Although the rate of change of distance with respect to $r$ becomes infinite, the distance from a point outside the event horizon to the event horizon is finite. Write a test program that integrates $d\sigma$ from a point $r = a$ outside the event horizon to a point arbitrarily close to the event horizon. What is the distance from $r = 4$ to $r = 2$ when $M = 1$?

**Exercise 18.9.** Event horizon

a. Although general relativity predicts the shape of orbits around any spherically symmetric gravitational object, not all objects have an event horizon. (The objects that do are black holes.) The event horizon assumes that the mass of the entire object is within the horizon, which implies very high mass densities. Calculate the $r$ coordinate of the event horizon for an object having the mass of the Earth and compare it to the radius of the Earth. Repeat the calculation for the Sun.

b. The event horizon for the black hole believed to exist at the center of our galaxy has an event horizon of $r = 7.6 \times 10^9$ m. What is its mass in units of our own sun (solar mass)?

The variable $t$ in the Schwarzschild metric is the time as measured by a faraway observer. Time as measured by a local observer is known as the proper time, $\tau$. Observers experience time as measured by their wristwatches and (18.19) shows that the wristwatch time interval $d\tau$ depends
on location. A faraway observer who measures the time between two light flashes records a value of $dt$, while an observer standing next to these flashes measures a time interval $d\tau$ given by

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2. \quad (18.24)$$

Proper time intervals near a gravitational mass are clearly smaller than faraway time intervals. This result gives rise to the gravitational red shift when applied to light.

**Exercise 18.10.** Local and faraway time

Estimate the time measured by an observer far from the Earth if an observer on the surface of the Earth measures one hour. Does special relativity play a role in an actual measurement?

### 18.5 Particle and Light Trajectories

The physics describing the trajectory of a particle in the vicinity of a gravitational mass can be formulated using the principle of stationary aging (see Hanc). This principle states that a particle takes a path through spacetime such that the elapsed time $\delta\tau$ recorded by the wristwatch attached to the particle is a maximum. (In general, $\delta\tau$ is an extremum so it also could be a minimum or a saddle point.) We construct a Lagrangian using the Schwarzschild metric such that

$$\tau = \int_{\text{initial event}}^{\text{final event}} d\tau = \int_{\text{initial event}}^{\text{final event}} \mathcal{L}(r, \dot{r}, \phi, \dot{\phi}) \, dt. \quad (18.25)$$

This Lagragian is given by

$$\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}) = \left[\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2 \dot{\phi}^2\right]^{1/2}, \quad (18.26)$$

Because the Lagrangian in (18.26) satisfies an extremum principle it must satisfy Lagrange’s equation, (18.15). We take the required derivatives and simplify terms to obtain the following system of first-order differential equations:

$$\frac{dr}{dt} = \dot{r} \quad (18.27a)$$

$$\frac{d\dot{r}}{dt} = \frac{4M^3 - 4M^2r - 4M^2r^3\dot{\phi}^2 + 4Mr^4\dot{\phi}^2 - r^5\dot{\phi}^2 + r^2(M - 3Mr^2)}{(2M - r)r^3} \quad (18.27b)$$

$$\frac{d\dot{\phi}}{dt} = \dot{\phi} \quad (18.27c)$$

$$\frac{d\dot{\phi}}{dt} = \frac{2(-3M + r)\dot{r}\dot{\phi}}{(2M - r)r} \quad (18.27d)$$

$$\frac{dt}{dt} = 1. \quad (18.27e)$$

Note that the independent variable in 18.27 is faraway time. The metric provides an additional differential equation if we wish to track the particle’s proper time, $\tau$.

$$\frac{d\tau}{dt} = \left[\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2 \dot{\phi}^2\right]^{1/2}. \quad (18.28)$$
Exercise 18.11. General relativistic trajectories

a. Write a program that plots the general relativistic trajectory of a particle using Schwarzschild coordinates. Verify that circular orbits are obtained for \( v = \sqrt{\frac{M}{r}} \) for \( r \geq 6M \).

b. Show that there are no stable circular orbits for \( r < 6M \).

c. Add the differential equation for proper time. What is the proper time for one complete orbit at \( r = 6 \)? This interval is the orbital period as measured by an observer traveling with the particle. Compare this wristwatch orbital period to the faraway orbital period and to the time interval predicted by (18.24). Explain any differences in your numerical values.

d. Perturb the circular orbit at \( r = 9 \) by giving the particle an initial tangential velocity of \( v = 0.345c \). At what rate does the perihelion of the orbit advance?

The equations for light can be obtained by adding a constraint to (18.26) using a Lagrange multiplier. The constraint is the condition that the proper time along a light worldline is zero.

\[
0 = -(1 - \frac{2M}{r}) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2.
\]  

(18.29)

If we add the Lagrange multiplier, do the differentiation, and simplify terms, we obtain rate equations that can be solved using standard numerical techniques. (The use of a computer algebra program would be helpful.)

\[
\frac{dr}{dt} = \dot{r}
\]

(18.30a)

\[
\frac{d\dot{r}}{dt} = -\frac{4M^2 + 2Mr + (r - 5M)r^3\dot{\phi}^2}{r^3}
\]

(18.30b)

\[
\frac{d\dot{\phi}}{dt} = \dot{\phi}
\]

(18.30c)

\[
\frac{d\cdot\phi}{dt} = \frac{2(-3M + r)r\dot{\phi}}{(2M - r)r}
\]

(18.30d)

\[
\frac{dt}{dt} = 1.
\]

(18.30e)

Exercise 18.12. Light trajectories

a. Write a program that plots the general relativistic trajectory of light using Schwarzschild coordinates. Demonstrate the deflection of star light passing near a gravitational mass by plotting the trajectory of a light ray.

b. Verify that light orbits a black hole at \( r = 3 \) and \( M = 1 \).

c. Show that a gravitational mass can act as a lens by plotting the trajectory of two light rays that leave a point source at different angles, pass on opposite sides of the mass, and then later cross. Do the two light beams always arrive at the crossing point at the same time?
18.6 Seeing

Because of the nonlinearity of the Schwarzschild metric, simulation plays an essential role. A calculation of a view of the stars in the vicinity of a black hole, for example, would require the solution of the light-ray trajectory for angles within the eye’s field of view.

The angles drawn on a Schwarzschild map are not the same as the angles seen by an observer because distances on the map are distorted. A stationary observer at a constant $r$-value is known as a shell observer because he is on a stationary shell at fixed $(r, \phi)$ coordinates. Launch angles measured by such a shell observer can easily be converted to angles on the Schwarzschild map by taking into account the contraction by $\sqrt{1 - 2M/r}$ in the radial direction.

$$\tan \theta_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} \tan \theta_{\text{schw}}.$$  \hspace{1cm} (18.31)

Use this transformation in Exercise 18.13 and Project 18.20.

Exercise 18.13. Knife-edge trajectory

Many important properties of light rays can be expressed in terms of an impact parameter, $b$, defined as

$$b = r \left(1 - \frac{2M}{r}\right)^{-1/2} \sin \theta_{\text{shell}}.$$  \hspace{1cm} (18.32)

For example, light that is launched with $b = \sqrt{2}\sqrt{\frac{M}{r}}$ enters an unstable orbit that teeters between an escape to infinity and a plunge into the black hole. This trajectory is known as a knife-edge trajectory because the result is very sensitive to the initial conditions and numerical roundoff error and cannot be predicted. What will a shell observer see if he looks into space at an angle that has this impact parameter?

Problem 18.14. Seeing near a black hole

Imagine a grid of light beacons located far away from a black hole in the $\phi = \pi$ direction. A shell observer at $\phi = 0$ with an arbitrary value of $r$ attempts to view the grid by looking toward the black hole. What will she see? One way to answer this question is to assume a reasonable field of view (for example, 180°) for the eye and calculate light rays leaving the eye at equal angular intervals. Compute the light paths and tabulate where the ray crosses the beacon grid as a function of angle. Because it is unlikely that the light rays will intersect a beacon location, use interpolation to determine the angles at which beacons appear. Plot these locations to show the observer’s view.

18.7 General Relativistic Dynamics

In general relativity, the magnitude of the angular momentum $L$ per unit mass $m$ of a particle is

$$\ell \equiv \frac{L}{m} = r^2 \frac{d\phi}{d\tau},$$  \hspace{1cm} (18.33)

and the energy $E$ per unit mass is

$$e \equiv \frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}.$$  \hspace{1cm} (18.34)
We can solve (18.33) for $d\phi$ and (18.34) for $dt$ and substitute the result into the time-like form of the metric and obtain a relation for $dr/d\tau$:

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - \left(1 - \frac{2M}{r}\right)\left[1 + \left(\frac{\ell}{r}\right)^2\right]. \tag{18.35}$$

In analogy with the classical effective potential function for a particle in a gravitational field, we use (18.35) to define a relativistic effective potential (see Figure 18.3):

$$\left(\frac{V(r)}{m}\right)^2 = \left(1 - \frac{2M}{r}\right)\left[1 + \left(\frac{\ell}{r}\right)^2\right]. \tag{18.36}$$

**Exercise 18.15.** Energy and angular momentum
Show that the energy and angular momentum are conserved for the orbits you observed in Exercise 18.11.

**Exercise 18.16.** Effective potential
Add a plot of the effective potential, $V(r)$, to your program for Exercise 18.11. Add a horizontal line showing the energy per unit mass and place a red marker on this line showing the particle’s radial position. Describe the effective potential and the motion of the marker when the orbit is circular, when the orbit precesses, and when the orbit plunges toward the event horizon.

### 18.8 The Kerr Metric

Because almost all astronomical objects rotate, most black holes likely have angular momentum. The metric for a spinning black hole was derived by Kerr in 1964. For simplicity, we show the metric for particle motion in the equatorial plane. Note that this metric contains a new angular momentum parameter, $a$:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4Ma^2}{r}dt\,d\phi - \left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right)^{-1} dr^2 - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right)r^2d\phi^2. \tag{18.37}$$
Because there are two values at which the coefficient of \(dr^2\) increases without limit, \(r_h = M \pm \sqrt{M^2 - a^2}\), there are two horizons. We also see that the largest real value of \(a\) consistent with real values of \(r_h\) is \(a = M\). This maximum value of \(a\) limits the angular momentum of a black hole. Because we are interested in maximizing the effect of rotation, we simplify (18.37) by letting the angular momentum parameter take on its maximum value. The metric for this extreme Kerr black hole is

\[
\frac{dr^2}{dt^2} = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4Ma}{r}dt\,d\phi - \left(1 - \frac{M}{r}\right)^{-2}dr^2 - R^2\,d\phi^2,
\]

where

\[
R^2 \equiv r^2 + M^2 + \frac{2M^3}{r}.
\]

We recast this metric as a Lagrangian and follow the derivation by Hanc and Tuleja, and obtain the rate:

\[
\frac{dr}{dt} = \dot{r},
\]

\[
\frac{d\dot{r}}{dt} = \frac{(M - r)^2(M - 2M^2\dot{\phi} + M^3\dot{\phi}^2 - r^3\dot{\phi}^2)}{r^4}
+ \frac{2M^3 - 2M^4\dot{\phi} + 3Mr^2 - M^2r(1 + 6r\dot{\phi})}{r^2(M - r)^2} \dot{r}^2,
\]

\[
\frac{d\phi}{dt} = \dot{\phi},
\]

\[
\frac{d\dot{\phi}}{dt} = \frac{4M^3\dot{\phi} - 2M^4\dot{\phi}^2 + 6Mr^2\dot{\phi} - 2r^3\dot{\phi} - 2M^2(1 + 3r^2\dot{\phi}^2)}{r^2(M - r)^2} \dot{r},
\]

\[
\frac{d\dot{\phi}}{dt} = 1.
\]

**Problem 18.17.** Falling into a spinning black hole

a. Write a program that plots the general relativistic trajectory of a particle near an extreme black hole using (18.40).

b. Follow the trajectory of a particle that starts from rest far from the center of the extreme black hole. Describe the trajectory.

c. A particle is thrown with an angular momentum opposite to the hole’s spin starting at \(r = 3M\). Write a program to simulate this situation and describe the particle’s motion.

A space ship near a black hole must fire its rockets radially to keep from falling into a black hole. It has an angular momentum appropriate for that radius so that the remote stars do not move overhead and therefore does not fire its rockets tangentially. However, if the space ship moves inward, it must fire its rockets tangentially or it will be swept sideways with respect to the remote stars. (The ship must only fire its rockets while moving inward.) This effect, known as *frame dragging*, occurs near any spinning gravitational object including Earth.
Problem 18.17 shows that frame dragging becomes dramatic as the falling particle approaches the horizon for the extreme black hole, $r_h = M$. (The horizon is where the metric coefficient of $dr^2$ becomes infinite.) Note that the coefficient of the $dt^2$ term goes to zero at $r = r_s = 2M$. This value is called the static limit. The space between the static limit and the horizon is dragged along in the direction of rotation of the black hole so that an observer cannot remain at a fixed angle no matter how powerful her rockets are.

18.9 Projects

Numerical relativity is still in its infancy, but is making progress in simulating astrophysical scenarios such as binary black hole mergers, binary neutron star mergers, and supernova core collapse. A key problem is achieving long-time stability of the numerical solutions. A search on “numerical relativity” will yield many interesting Web sites and entrees to current research.

Project 18.18. Three-dimensional rapidly moving objects

Extend the analysis in Section 18.1 to three-dimensional objects and model their appearance as seen by a single observer at the origin using the transformation and rendering techniques described in Chapter 17. Does a sphere appear to be a sphere even when it passes by an observer? Does a cube appear to be a cube?

Project 18.19. Light Links

a. Imagine two stationary observers near a black hole wishing to establish a communication link using a laser beam. In what direction should the laser be pointed to establish such a link? Simulate this scenario using two draggable objects on a Schwarzschild map and draw the light ray representing the communication link. Use a root finding algorithm, such as the bisection method introduced in Chapter 6, to determine the proper launch angle. Calculate and display the proper distance along this light path and study how this distance changes as the light path grazes the event horizon.

b. Construct a light triangle connecting three observers. Display the sum of the interior angles as measured by the observers to simulate Gauss’s mountain top experiment.

Project 18.20. Seeing orbits

Viewing an orbit requires that we calculate the particle’s trajectory and the trajectory of the light ray from the particle to the viewer. An added complication arises because the light reaching the view is retarded by the travel time. Write a program that shows an orbiting particle as seen by a stationary observer in the equatorial plane by keeping track of both particle and light-link parameters.

18.10 References


James Terrell, “Invisibility of the Lorentz contraction,” Phys. Rev. 116, 1041–1045 (1959). This paper corrected the erroneous belief that that had been taught for fifty years that an observer sees the Lorentz contraction when viewing a relativistically moving object.


Most general relativity texts begin with a treatment of tensor analysis. The following two texts present this material using the four-dimensional spacetime metric.


The Web site, <http://archive.ncsa.uiuc.edu/Cyberia/NumRel/NumRelHome.html>, developed by the National Center for Supercomputing Applications, is one of many that discuss Einstein’s contributions and recent progress in numerical relativity.