Student Understanding Of The Physics And Mathematics Of Process Variables In *P-V* Diagrams

Evan B. Pollock,[1] John R. Thompson,[1,2] and Donald B. Mountcastle[1]

[1]*Department of Physics and Astronomy and* [2]*Center for Science and Mathematics Education Research*
University of Maine, Orono, ME

Abstract. Students in an upper-level thermal physics course were asked to compare quantities related to the First Law of Thermodynamics along with similar mathematical questions devoid of all physical context. We report on a comparison of student responses to physics questions involving interpretation of ideal gas processes on *P-V* diagrams and to analogous mathematical qualitative questions about the signs of and comparisons between the magnitudes of various integrals. Student performance on individual questions combined with performance on the paired questions shows evidence of isolated understanding of physics and mathematics. Some difficulties are addressed by instruction.

INTRODUCTION

As part of ongoing research at the University of Maine (UMaine) we are exploring student learning in upper-level thermal physics courses, primarily taken by physics majors. There exists a limited body of research on this topic [1-6]. Prior work has suggested that particular difficulties experienced at the introductory level are also evident at the advanced undergraduate level [1]. Previous results on the teaching and learning of the First Law have documented several common difficulties, such as indiscriminate application of the concept of a state function [2,3].

One theme of our research is the extent to which student mathematical conceptual difficulties may affect understanding of physics concepts in thermodynamics. We have presented results of this nature for partial derivatives and their relationship to material properties and the Maxwell relations [4,6]. Our purpose for this study is to determine if student difficulties with mathematical concepts are impacting students' abilities to solve problems related to the First Law of Thermodynamics. In this paper we describe studies on energy transfers in and out of a system.

Mathematics is a vital part of solving many physics problems. It is a universal language that can often condense a complicated conceptual problem into a simple relationship between variables. Many convenient representational tools exist in physics, e.g., equations, graphs and free-body diagrams, that simplify analysis of a complex problem. Appropriate interpretation of these representations requires recognition of the connections between the physics and the mathematics built into the representation and subsequent application of the related mathematical concepts [7].

In thermodynamics, two- or three-dimensional graphical representations of physical processes are especially useful in helping to understand these processes. These diagrams can contain information about the thermodynamic "path" followed in a process, regions of different phase, and critical behavior. *P-V* diagrams are used extensively as representations of physical processes as well as of the corresponding mathematical models. Information can quickly and easily be obtained from this representation. We focus on student understanding of *P-V* diagrams in the context of determining work related to the First Law of Thermodynamics.

Meltzer developed a set of questions probing student understanding of the First Law and related quantities based on processes shown on a *P-V* diagram [3]. In addition to replicating Meltzer's experiment in our upper-level thermodynamics course, we wanted to know if some of the conceptual difficulties might originate in the mathematics. We designed questions that are analogous to Meltzer's questions, but purely mathematical in nature – all physics is removed from the setting. The math context involves qualitative

questions regarding comparisons and determinations of magnitudes and signs of integrals.

We present results based on three years of data from UMaine's upper-level *Physical Thermodynamics* course, taught in the fall semesters of 2004-2006 (by DBM). UMaine's introductory calculus-based physics sequence does not include thermal physics; for some students this was their first exposure to the topic. All students in the data set had completed three semesters of calculus, including multivariable calculus, and one or more courses in ordinary differential equations.

PHYSICS QUESTION: WORK COMPARISON

Meltzer asked students to compare energy transfers and changes (work, heat transfer, internal energy changes) for two ideal gas processes (#1 and #2), shown on a *P-V* diagram, that both begin and end in the same states (A and B) (Fig. 1). The first problem, P1, asks the students to compare the works done by the gas in the two processes. The correct response for P1 can be obtained by recognizing that the work done by the gas is $\int P dV$, and that this integral is represented by the area under the curve for that process.

Meltzer found that between 20% and 30% of students incorrectly stated that the works done by the gas over both paths were equal. Common student reasoning used the idea that work was path independent, and/or that the work only depended on the endpoints in the diagram. Meltzer attributed this to one of two reasons: first, a (reasonable) overgeneralization of work in the context of conservative forces in mechanics to thermodynamics, and second, a conceptual difficulty distinguishing between heat, work, and internal energy. Meltzer's conclusion was that students were inappropriately attributing state function properties to work.

We asked these physics questions prior to explicit instruction on the First Law and *P-V* diagrams in 2004 and 2005 (*N*=15), but after instruction in 2006 (*N*=6). The pretest performance on P1 at UMaine was generally consistent with, although worse than, Meltzer's in his post-instruction introductory courses: of 15 students, 5 (33%) gave the correct comparison, while 8 (53%) stated that the works were equal. Students with correct responses used "area under the curve" or "$\int P dV$" in their explanations. Students justified equal works using mostly state function-like reasoning: "Work done is independent of path"; "Both processes start and end in same place"; "Both processes have same beginning and end points."

Students did much better when given the question as a post-test: 5 of 6 students gave correct responses, and none said that the works were equal. Correct

explanations included explicit references to the integral, to area under the curve, and to the path dependence of work.
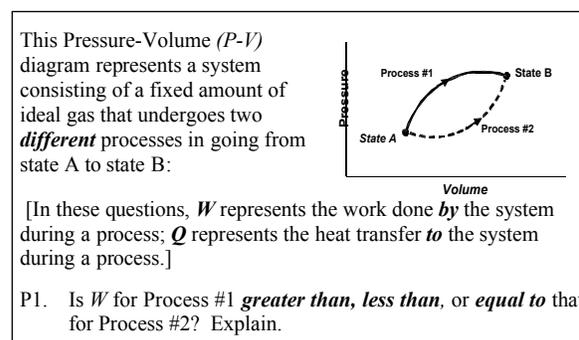
This Pressure-Volume *(P-V)* diagram represents a system consisting of a fixed amount of ideal gas that undergoes two *different* processes in going from state A to state B:

[In these questions, *W* represents the work done *by* the system during a process; *Q* represents the heat transfer *to* the system during a process.]

P1. Is *W* for Process #1 *greater than, less than,* or *equal to* that for Process #2? Explain.

FIGURE 1. Question P1. (From Ref. 3.)

MATHEMATICS QUESTION: INTEGRAL COMPARISON

As mentioned, Meltzer's work question (P1) was paired with an analogous mathematics question (M1) designed to elicit any underlying mathematical difficulties in this context. In M1, we asked students about the magnitudes of two integrals that have identical beginning and end points (M1). The two processes in P1 are replaced with two different curves. We asked the math question before the physics question, usually in the first week of class.

Two versions of M1 were given, to different students (see Figs. 2 and 3). In Version 1 (V1), the curves are distinguished with labels *Path 1* and *Path 2*. The integrals are expressed using the same variable, *z,* for both paths. Because the two curves were represented using the same variable, this version is a mathematical isomorph of P1, in which the two processes are both considered graphs of pressure vs. volume (*P* vs. *V*). In Version 2 (V2), the curves are labeled as two different functions, *f(y)* and *g(y)*; the integrals also reflect this distinction. In V2 the notation of the two curves is more mathematically rigorous; the question is analogous to P1 rather than isomorphic.

The results for M1 (see Table 1) lend insight into student conceptual difficulties related to integration. Approximately one-quarter of the students (5/21) stated that the two integrals were equal in magnitude, regardless of which version they were given. The reasoning given for these responses were often identical to that given for P1: "The function does not depend on the path, but only on the end points"; "same endpoints"; "both have same end points and equal functions." These explanations of integral path independence are consistent with treatment of both

integrals as antiderivatives rather than as Riemann sums.

Furthermore, the data from the two versions of question M1 suggest that students treat the versions differently. A much higher fraction of students (77% vs. 38%) gave correct responses in V2, when the different functions were labeled as such, rather than in V1, where they were labeled as the same variable, with only the path to distinguish them.
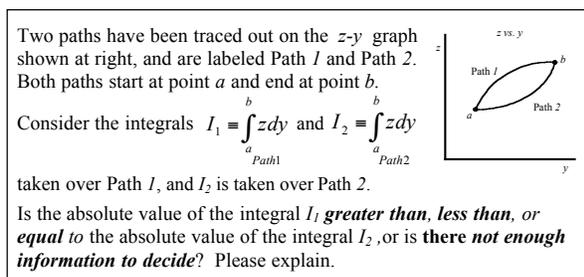
Two paths have been traced out on the *z-y* graph shown at right, and are labeled Path *1* and Path *2*. Both paths start at point *a* and end at point *b*.

Consider the integrals $I_1 \equiv \int_a^b zdy$ and $I_2 \equiv \int_a^b zdy$ taken over Path *1*, and I_2 is taken over Path *2*.

Is the absolute value of the integral I_1 *greater than*, *less than*, or *equal* to the absolute value of the integral I_2, or is there *not enough information to decide*? Please explain.

FIGURE 2. Question M1 Version 1 (V1), a mathematical analog of P1.

Two functions have been graphed on the *z-y* graph shown at right, and are labeled *f(y)* and *g(y)*. Both functions start at point *a* and end at point *b*.

Consider the integrals

$$I_1 \equiv \int_a^b f(y)dy \quad \text{and} \quad I_2 \equiv \int_a^b g(y)dy$$

Is the absolute value of integral I_1 *greater than*, *less than or equal* to the absolute value of integral I_2, or is there *not enough information to decide*? Please explain.
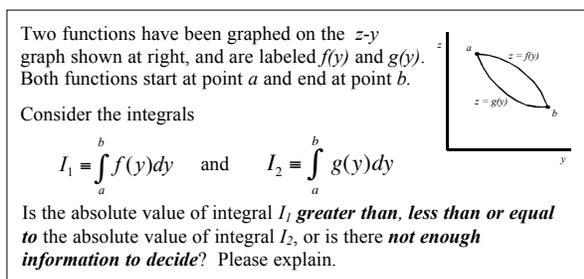
FIGURE 3. Question M1 Version 2 (V2), a mathematical analog of P1.

TABLE 1. Student responses to M1, overall and by version.

I_1 vs. I_2	All (N=21)	V1(N=8)	V2 (N=13)
Greater Than	**13 (62%)**	**3 (38%)**	**10 (77%)**
Not Enough Information	1 (5%)	1 (12%)	---
Equal	5 (24%)	3 (38%)	2 (15%)
Blank	1 (5%)	---	1 (8%)
Less Than	1 (5%)	1 (12%)	---

ANALYSIS OF MATHEMATICS-PHYSICS RESPONSE PAIRS

A comparison of individual student responses to the question pair of P1 and M1 (Table 2) provides a stronger indication of the mathematical or physical nature of the student difficulty. On the pretest, one third answered both questions correctly, and one third stated that both the works and integrals were equal. On the post-test, 5 of 6 were correct on both questions. We enumerate notable points that the data uncover.

1. All of the students who answered the physics question correctly also answered the math question correctly.
2. All of the students who answered the math question incorrectly also answered the physics question incorrectly.
3. Of the students who said the works were equal, more than half said the integrals were also equal.
4. None of the students in the *other* category answered either question correctly.

TABLE 2. Student paired responses to M1 (combined V1 and V2) and P1. Two classes answered the physics questions before instruction, one class after instruction.

M1	P1	Physics Pre (N=15)	Physics Post (N=6)
$I_1 > I_2$	**$W_1 > W_2$**	**5 (33%)**	**5 (83%)**
$I_1 > I_2$	$W_1 = W_2$	3 (20%)	—
$I_1 = I_2$	$W_1 = W_2$	5 (33%)	—
other		2 (13%)	1 (17%)

Analysis of the data suggests the following interpretations. Students who answered that the math integrals were equal and that the works were equal demonstrate, to us, a math difficulty as well as a possible physics difficulty. The all-or-nothing nature of the responses (points 1, 2, 4 above) suggests that correct understanding of the mathematics is necessary in order to answer the physics question correctly.

As point 3 implies, less than half of the erroneous work comparisons were paired with correct integral comparisons; these students had genuine conceptual difficulties with the physics while possessing an understanding of the math. Two of the three students in the M1-correct/works-equal category (second row of Table 2) used *area under the curve* reasoning to compare integrals in M1 but then used *path independence* or *state function* reasoning to equate works in P1. These students are inappropriately invoking *state function* reasoning in the physical context of work, as described by Meltzer and by Loverude et al. [2,3]; however, these students do not use this reasoning in the pure mathematical context.

The high post-test performance implies that instruction may successfully address this issue.

DISCUSSION AND CONCLUSIONS

We have three components to this research that we summarize and discuss here. First, we asked questions comparing work done based on paths on a *P-V* diagram. Our pretest results on upper-level students were consistent with the original results obtained after

instruction at the introductory or intermediate level [3], namely that a significant fraction of students are attributing the property of path independence to thermodynamic work. We also see much better results obtained after instruction at the upper level, implying that this issue may be a matter of familiarity with the material.

Second, we asked two different versions of a mathematics question analogous to the work comparison described above. The results show that it is not trivial for students to compare integrals of functions with the same endpoints but that follow different paths (curves). The notation used to describe, and to distinguish between, the functions seems to affect student performance: labeling two different curves using the same variable leads to more students equating integrals.

This second result is directly relevant to thermodynamics, in which one variable often represents multiple functions (e.g., *P(V)* over two different paths). It is arguably necessary to use less mathematically rigorous notation in physics to keep the physical meaning. However, what might be considered a trivial distinction in notation to physicists seems to make a substantial impact on even advanced students' interpretations of the diagrams.

Third, we checked individual student responses on the paired physics and math questions. We found that some of the difficulties that arise when comparing thermodynamic work based on a *P-V* diagram – e.g., treating work as path independent – may be attributed to difficulties with the *mathematical* aspect of the diagram, in particular with the correct application of an understanding of integrals, rather than simply physics conceptual difficulties (e.g., treating work as a state function). These results suggest that some students aren't attributing state function properties to work so much as failing to recognize the same variable as two different functions during integration.

In related work, we have also administered and analyzed paired questions for Meltzer's internal energy comparison and a mathematical analog (i.e., integrating a path-*independent* quantity). (Space constraints prevent a full description.) Performance on the physics question is very high (>90%); students typically invoke the state function property of internal energy. The math question performance is much worse. The difference in outcomes on these questions shows that some students lack an understanding of the mathematical properties of state functions, and only rely on the physics and/or the physical aspects of the system in question. These results suggest that while students have a better understanding of *state function* than of *process function* in physics, the reverse is true for the mathematical distinction between these two concepts. The extent to which this may be related to the calculus concepts of *antiderivative* and *Riemann sum*, or to the larger grain-sized distinction between function and variable, is yet to be determined.

Finally, while this analysis does not endorse any theoretical framework, an analysis of the data through the lens of resource activation may be enlightening. The idea of "compelling visual attribute" [8] could provide alternate explanations of two of our results. First, for P1, the endpoints of the *P-V* diagrams may tend to make students focus only on the two states represented, rather than the process followed between those states. Second, M1 contains a diagram and symbolic expressions of the integrals; for V1, these symbolic expressions differ only in the path label. Perhaps when presented with both symbolic versions of integrals and a graphical representation of the functions, some students may preferentially cue on the symbolic version to decide that integrals of identical variable expressions, with identical limits, but over different paths, are equal.

Future work

The data set contains additional question pairs that add to this work. We have also collected homework sets and exams from this population. In addition, a more in-depth probing of student understanding of the state function concept is warranted, including the relationship to calculus concepts.

ACKNOWLEDGMENTS

REFERENCES

1. D.E. Meltzer, *2004 Phys. Educ. Res. Conf.*, edited by J. Marx *et al*., AIP Conf. Proc. **790**, 31-34 (2005).
2. M.E. Loverude, C.H. Kautz, and P.R.L. Heron, *Am. J. Phys.* **70**, 137-148 (2002).
3. D.E. Meltzer, *Am. J. Phys*. **72**, 1432-1446 (2004).
4. J.R. Thompson, B.R. Bucy, D.B. Mountcastle, *2005 Phys. Educ. Res. Conf.,* edited by P. Heron *et al.,* AIP Conf. Proc. **818**, 77-80 (2006).
5. M.J. Cochran and P.R.L. Heron, *Am. J. Phys*. **74,** 734-741 (2006).
6. B.R. Bucy, J.R. Thompson and D.B. Mountcastle, *2006 Phys. Educ. Res. Conf.,* edited by L. McCullough *et al.,* AIP Conf. Proc. **883**, 157-160 (2007).
7. E.F. Redish, in Proceedings of *World View on Physics Education in 2005: Focusing on Change,* Delhi, 2005 (2005). Available at http://www.physics.umd.edu/perg/ papers/redish/IndiaMath.pdf.
8. A. Elby, *J. Math. Behav.*, **19**, 481-502 (2000).