Symbols: Weapons of Math Destruction

Eugene Torigoe and Gary Gladding

Department of Physics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

Abstract.
This paper is part of an ongoing investigation of how students use and understand mathematics in introductory physics. Our previous research [1] revealed that differences in score as large as 50% can be observed between numeric and symbolic versions of the same question. We have expanded our study of numeric and symbolic differences to include 10 pairs of questions on a calculus based introductory physics final exam. We find that not all physics problems exhibit such large differences and that in the cases where a large difference is observed that the largest difference occurs for the poorest students. With these 10 questions we have been able to develop phenomenological categories to characterize the properties of each of the questions. We will discuss what question properties are necessary to observe differences in score on the numeric and symbolic versions. We will also discuss what insights these categories give us about how students think about and use symbols in physics.

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INTRODUCTION

We have been investigating how students use and understand mathematics in introductory physics classes. Our interest in mathematics is related to how students use and understand the meaning of symbols rather than any difficulties they may have with algorithmic mathematical activities like solving two equations and two unknowns. Our previous research [1] revealed that differences in score as large as 50% can be observed between numeric and symbolic versions of the same question. While we were able to measure large effects, the scope of the study was limited because we only studied two questions. The two questions used in the previous study (each with a numeric and a symbolic version) dealt with one dimensional kinematics and the scores for the numeric version of both questions were high. We have expanded our study of numeric and symbolic differences to include 10 pairs of questions that span many different introductory mechanics topics and have a variety of difficulty levels.

METHODOLOGY

The subjects of the study were students who were enrolled in the calculus-based introductory mechanics course, Physics 211, at the University of Illinois, Urbana-Champaign in the spring 2007 semester. Students in Physics 211 take three multiple-choice midterm exams and a cumulative final. While numeric questions are the most common type of question, symbolic questions are not uncommon. There were 765 students who completed one of the two randomly administered versions of the final exam. Each version of the final contained either the numeric or symbolic version of each of the 10 questions.

All but one of the 10 paired questions contained analogous choices for each the numeric and symbolic versions of the question[1]. To discourage cheating many of the choices were rearranged between the versions.

To test the equivalence of the groups we compared the average midterm exam score of the students for each of the final exam versions. The average midterm score and standard error for final 1 was 78.9±0.5, and for final 2 was 78.6±0.5. Based on their average midterm exam scores the two groups are equivalent.

RESULTS AND ANALYSIS

The numeric/symbolic difference by group

Table 1 describes the scores observed on the numeric and symbolic versions of each question. The errors shown represent the standard deviation of the mean. While there exist large differences between numeric and symbolic scores for some questions, there are other questions that show no significant difference. To further study the relationship between the differences between numeric and symbolic questions and overall course performance we divided the class into three subgroups based on the total course points. The groups are the bottom

[1] The incorrect choices for the two versions of Question 4 were similar but with *i* and negative signs omitted from the choices from the numeric version.

TABLE 1. Average and standard error for numeric and symbolic versions of each question in the study

	Q1	Q2	Q3	Q4	Q5
Numeric	91.5 ± 1.4	93.3 ± 1.3	79.6 ± 2.1	90.5 ± 1.5	44.9 ± 2.5
Symbolic	70.4 ± 2.3	56.8 ± 2.5	63.4 ± 2.4	82.3 ± 1.9	31.9 ± 2.3
	Q6	**Q7**	**Q8**	**Q9**	**Q10**
Numeric	61.2 ± 2.4	76.0 ± 2.1	33.2 ± 2.3	78.6 ± 2.0	48.8 ± 2.3
Symbolic	52.9 ± 2.5	75.6 ± 2.1	29.8 ± 2.2	54.5 ± 2.5	52.8 ± 2.4

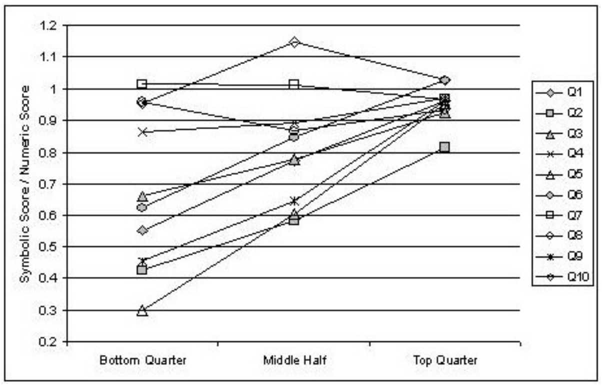

FIGURE 1. The ratio of the symbolic score over the numeric score for each question and for three groups based on total points in Physics 211

quarter, the middle half and the top quarter[2]. For each group and for each question in the study we calculated the ratio of the average score on the symbolic version to the average score on the numeric version. We interpret this ratio to represent the likelihood that the students who could solve the numeric version correctly would also solve the symbolic version correctly. Even though no individual was given both versions of a single question, we believe we are justified in this interpretation because of the equivalence of the groups. The results of this analysis are shown in Figure 1. In the cases where we observed a large difference between the numeric and symbolic versions, the largest difference (smallest ratio) was observed for the bottom quarter of the class. The top quarter of the class, by comparison, rarely showed any distinction between the numeric and symbolic versions. For the top quarter only one question had a ratio lower than 0.93. This correlation suggests that there are differences in the ways top students and bottom students understand symbols in physics.

[2] The groups were created in this manner because of our interest in the issue of retention. The groups distinguish students who fail or who are on verge of failing (bottom quarter) with those that pass comfortably (middle half) and those that excel (top quarter).

Phenomenological Categories

We performed an analysis involving a careful inspection of each of the 10 questions including an inspection of the popularity of the correct choices, popularity of the incorrect choices, and an analysis of students written work to create a coding scheme to differentiate the questions. We have identified the following properties based on our analysis.

Difficulty for the top quarter The average for the numeric version of the question.

Multiple equations This code distinguishes whether the problem is commonly solved with one equation or with multiple equations

Coupled The solution is coupled if, for example, there are two equations and two unknowns and both unknowns are present in both equations. This is contrast to a sequential problem where with two equations and two unknowns there is an equation where only one of the unknowns is present.

General equation manipulation This code signifies whether it is possible to obtain one of the incorrect choices by combining general equations or manipulating a single general equation with minimal changes (e.g. replacing x with d).

Specification of a slot variable A slot variable is a symbol found in a general equation. This code signifies that in order to reach the correct symbolic solution the student must replace a slot variable with a compound expression (e.g. replacing the slot variable v with a more specific compound expression $v/2$).

Table 2 shows the codes for each of the 10 questions in this study, as well as the measurement of the ratio of the average score on the symbolic version to that of the numeric version for the bottom quarter.

Separating Physics Difficulties from Math Difficulties

Because we are primarily interested in determining the role of mathematical difficulties we would like to be able to control for conceptual physics difficulties. We

TABLE 2. Coding of question properties for each of the 10 questions in the study

	Difficulty for the top quarter	**Multiple equations**	**Coupled**	**General equation manipulation**	**Specification of a slot variable**	**Ratio of the symbolic score to the numeric score for the bottom quarter**
Q.1	0.96	Y	N	Y	N	0.55 ± 0.06
Q.2	0.99	Y	N	Y	Y	0.43 ± 0.06
Q.3	0.97	Y	N	Y	N	0.66 ± 0.10
Q.4	0.98	N	N	N	Y	0.86 ± 0.08
Q.5	0.67	Y	N	N	Y	0.30 ± 0.12
Q.6	0.87	Y	N	N	Y	0.62 ± 0.13
Q.7	0.94	N	N	N	Y	1.01 ± 0.11
Q.8	0.49	Y	N	Y	Y	0.96 ± 0.23
Q.9	0.94	Y	N	Y	Y	0.45 ± 0.08
Q.10	0.82	Y	Y	N	N	0.95 ± 0.19

used the difficulty for the top quarter as a measure of the physics difficulty. Two questions (Questions 5 and 8) appear to be significantly more difficult for the top students. Consequently we removed them from further analysis in this study. This decision is supported by the fact that the average scores for the numeric version for the bottom quarter for each of these questions is 23%, consistent with the random guessing rate of 20%.

No numeric and symbolic difference

From Table 2 one can see that there are three remaining questions (Questions 4, 7, and 10) in which there was virtually no difference between the numeric and symbolic versions for the three groups in the class. Questions 4 and 7 were the only questions in which the solution could be found with a single equation. Further the solution to Question 7 depended mainly on the concept that linear momentum is conserved after a collision that caused rotation.

Students also showed no distinction between the numeric and symbolic versions of Question 10. This question is the only question in our sample involving coupled equations. In order to solve a coupled physics problem one has to setup both equations before one can proceed to the solution. For this reason, we expect that students will be forced to use the same procedure to solve both the numeric and symbolic versions of the question. The results are consistent with out expectation that there would be no difference between the versions.

Large numeric and symbolic difference

For the five remaining questions in our study (Questions 1, 2, 3, 6, and 9) it is apparent that there are many students who can solve a numeric question but who are unlikely to be able to solve the analogous symbolic version. From the data on Table 2 we can see commonalities between the questions for which a large difference is observed.

All five questions require that multiple equations be used to reach the correct solution. All of the questions can also be solved sequentially which we believe aids in the solution of numeric problems. When working with numbers in a sequential problem, the problem can be broken up into well defined steps during which an equation condenses into a numeric value. Unlike numeric solutions, symbolic solutions do not condense so compactly and often become more complex.

For all but Question 6 an incorrect choice can be obtained by the blind manipulation of general equations. For Questions 1, 2, and 3 the most popular incorrect answer is the choice (or one of the choices in the case of Question 2) that can be obtained by manipulating general equations.

Questions 2, 6, and 9 require that students specify a slot variable as a compound expression. For Question 2 the second most popular incorrect choice corresponded to using v instead of $v/2$ or t instead of $t/2$[3]. For Question 6 the most popular incorrect choice corresponded to a failure to specify the slot variable M. Question 9 (See Figure 2) required that students specify both the slot variable m as $3m$ and the slot variable v as $v/3$. Interestingly very few students failed to specify both slot variables, the most popular incorrect choices corresponded to a failure to specify either one or the other.

The findings of the preceding analysis can be summarized by the following statements: We see no difference in the average scores of symbolic and numeric versions if the correct solution to the problem involves the use of either (i) a single equation or (ii) two coupled equations. We do see significant differences in the average scores of symbolic and numeric versions if (i) the correct solution requires the use of multiple equations, none of which are coupled, or (ii) the correct solution requires a slot vari-

[3] There two different common ways of reaching the solution.

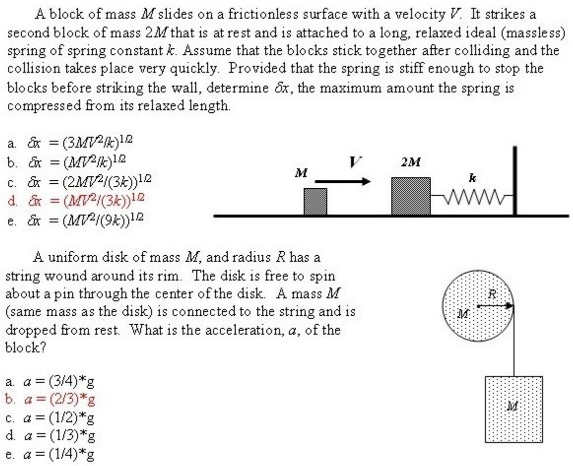

FIGURE 2. The symbolic versions of Question 9 (upper) and Question 10 (lower)

able to be replaced by a compound expression, or (iii) an incorrect choice can be obtained by manipulating general equations with no substantive replacements.

DISCUSSION

Data on 10 different numeric and symbolic pairs of questions that spanned difficulty level and topic have allowed us to develop a set of phenomenological categories to distinguish the questions. We have been able to show correlations between these categories and the differential performance on the numeric and symbolic versions.

A common indicator of difficulty for a symbolic question related to whether or not one of the incorrect choices could be obtained by the blind manipulation of general equations. This error could indicate that some students approach symbolic problems as a manipulation exercise. This may reflect a difference in the way they frame the activity of solving a symbolic problem to the activity of solving a numeric problem[2]. The framing of the exercise determines what resources are activated and what activities (games) are appropriate. This framing argument may also explain errors associated with the failure to specify slot variables, which is a form of blind equation manipulation.

An alternative explanation may be the added cognitive demand placed on the students when they solve symbolic problems[3]. Cognitive load theory posits the existence of the working memory and the long term memory. While long term memory is boundless, the working memory is easily overloaded. As discussed in an earlier section the activities related to sequential problems are very different for numeric and symbolic problems. Numbers allow for the alleviation of cognitive load because each step condenses into a numeric value, whereas each step in a symbolic problem ends with a potentially complex symbolic expression. Because symbols are carried from step to step, it is more difficult to clearly demarcate different steps.

Notational differences between numeric and symbolic problem solutions may also have a significant effect on cognitive load. There are two distinctions that are explicit in numeric solutions, but which are implicit in symbolic solutions. First, there is no explicit notation when solving symbolic problems to distinguish known from unknown symbols. Second, there is no notational difference between slot variables and specific known or unknown quantities[4]. In numeric problems there is a clear demarcation between a slot variable and a specific known (a number), as well as between an unknown (symbol) and a known (a number). When working on a symbolic problem these factors contribute to the cognitive load. Some students who fail to specify the symbols in the general equation may have an overly general sense of the meaning of symbols. Students may fail to specify slot variables because they may still consider them in some cases to be a general symbol. This could also be the case when students manipulate general equations and get an incorrect answer, they may think that the symbol is so general that even the object association of the symbol is variable.

CONCLUSION

By studying a variety of physics problems with both numeric and symbolic versions we have been able to develop a phenomenological coding to distinguish the cases where differences between numeric and symbolic versions of a physics question exists.

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[4] Sometimes subscripts can help with this distinction, but our observations find that few students use them