

# Comparing Student Use of Mathematical and Physical Vector Representations

Joel Van Deventer\* and Michael C. Wittmann\*<sup>†</sup>

\*Center for Science and Mathematics Education Research, University of Maine, Orono, ME 04469, USA

<sup>†</sup>Department of Physics and Astronomy, University of Maine, Orono, ME 04469, USA

<sup>†</sup>College of Education and Human Development, University of Maine, Orono, ME 04469, USA

**Abstract.** Research has shown that students have difficulties with vectors in college introductory physics courses and high school physics courses; furthermore, students have been shown to perform worse on a vector task with a physical context when compared to the same task in a mathematical context. We have used these results to design isomorphic mathematics and physics free-response vector test questions to evaluate student understanding of vectors in both contexts. To validate our test, we carried out task-based interviews with introductory physics students. We used our results to develop a multiple-choice version of the vector test which was then administered to introductory physics students. We report on our test, giving examples of questions and preliminary findings.

**Keywords:** Physics Education Research, Vectors

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## INTRODUCTION

Students in introductory physics classes have difficulties with vector tasks [1,2]. Some topics of difficulty include: vector magnitude, direction, addition, subtraction, dot product, cross product, and unit vectors. Furthermore, students have difficulty with the vector nature of physics concepts [3,4]. Recently, Shaffer & McDermott [5] observed introductory physics students performing better on a mathematics vector task when compared to performance on an isomorphic physics vector task. Their research has led to our current research focus of comparing student performance on isomorphic math/physics vector tasks. The following are our guiding research questions:

- 1) How does student performance on the isomorphic vector tasks compare?
- 2) What tools<sup>1</sup> are students using in math and physics contexts to accomplish vector tasks?
- 3) Are students transferring scalar and vector tools from mathematics to physics contexts in introductory physics courses?
- 4) What tools are missing in order to be successful?

Our limited discussion in this paper primarily focuses on questions one and two above.

<sup>1</sup>See [6] for specific meaning and background for the term tool.

## METHODS

We developed two isomorphic free response vector tests using equivalent questions in different contexts, one math and the other physics. However, two questions on the physics version have added difficulty and break strict isomorphism. Our tests were validated using interviews with early versions of the test. Using interview data and results from other researchers, we developed a multiple-choice version of each test. This tests was administered to an introductory calculus-based physics course.

The interview sample consisted of 10 self-selected students from a 1<sup>st</sup> semester calculus-based introductory physics course. Interviews were conducted near the end of the semester, when all students had used vectors in many different physics contexts.

The multiple-choice (MC) tests were administered to 28 students in a summer-session introductory calculus-based physics course at the end of the semester. A self-reported survey of the class indicates 83% (24 students) studied vectors in at least a prior high school physics course and half of those (13 students) also studied vectors in at least a high school math course. Both the interview sample and the MC sample were taught by the same instructor and covered the same material.

## Test Design

Influence from previous research [1-5] led to the development of preliminary free-response questions on graphical and algebraic representations of vectors. For example, we modified a one-dimensional graphical subtraction question from Shaffer & McDermott's research [5], as shown in Figure 1. An example of a MC question on an algebraic expression for a  $x$ -component is given in Figure 2. Two test questions are not strictly isomorphic, but contain added difficulty in the physics version. For example, the angle on the  $x$ -component algebraic expression question (Fig. 2) is labeled differently between contexts. Question ordering is the same on each version; however, the choices are randomized.

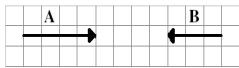
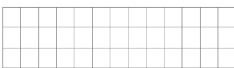
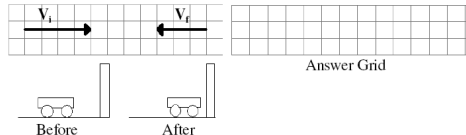
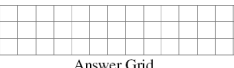
Math Version	
2. Consider vectors <b>A</b> and <b>B</b> below. Find <b>R</b> , the difference vector, when <b>A</b> is subtracted from <b>B</b> (i.e. $\mathbf{R} = \mathbf{B} - \mathbf{A}$ ).	
	 Answer Grid
Physics Version	
2. Consider a cart colliding into a wall. Its initial velocity $\mathbf{V}_i$ and the final velocity $\mathbf{V}_f$ are given as depicted below. In the space provided draw and label the change in velocity of the cart, $\Delta\mathbf{V}$ , where $\Delta\mathbf{V}$ is $\mathbf{V}_i$ subtracted from $\mathbf{V}_f$ (i.e. $\Delta\mathbf{V} = \mathbf{V}_f - \mathbf{V}_i$ ).	
	 Answer Grid

Figure 1. One-dimensional vector subtraction questions.

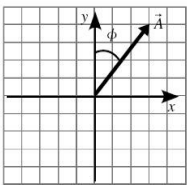
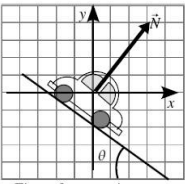
Math Version	
8. Which of the following algebraic expressions, if any, gives the accurate <u>magnitude</u> of the <u><math>x</math>-component</u> of vector $\vec{A}$ above, (i. e., $ \vec{A}_x $ ) ?	
(a) $ \vec{A}_x  =  \vec{A}  \tan \phi$ (b) $ \vec{A}_x  =  \vec{A}  \cos \phi$ (c) $ \vec{A}_x  = \frac{\sin \phi}{ \vec{A} }$ (d) $ \vec{A}_x  =  \vec{A}  \sin \phi$ (e) $ \vec{A}_x  =  \vec{A}  -  \vec{A}_y $ (f) $ \vec{A}_x  = \frac{\cos \phi}{ \vec{A} }$ (g) None of the above	 Figure from previous page
Physics Version	
8. Which of the following algebraic expressions, if any, gives the accurate <u>magnitude</u> of the <u><math>x</math>-component</u> $ \vec{N}_x $ ?	
(a) $ \vec{N}_x  =  \vec{N}  \sin \theta$ (b) $ \vec{N}_x  = \frac{\sin \theta}{ \vec{N} }$ (c) $ \vec{N}_x  = \frac{\cos \theta}{ \vec{N} }$ (d) $ \vec{N}_x  =  \vec{N}  -  \vec{N}_y $ (e) $ \vec{N}_x  =  \vec{N}  \cos \theta$ (f) $ \vec{N}_x  =  \vec{N}  \tan \theta$ (g) None of the above	 Figure from previous page

Figure 2. MC  $x$ -component algebraic expression questions.

The preliminary interviews were conducted by giving the math version of the vector free-response test first, followed by the physics version. Interviews were videotaped and students were asked to “think out loud” when working. Common incorrect responses given during the interviews and found in the literature [1] were used as distractors in the multiple-choice versions of the tests.

The MC tests were administered in the same manner as the interviews. Students completed and submitted the math version before receiving the physics version.

In the interests of shortening the MC test to a manageable length, while still remaining useful for instructors, we dropped several questions. For example, we dropped questions dealing with graphical magnitude and direction of vectors because students performed very well on these questions during interviews (60-100% correct). Our results are consistent with Nguyen & Meltzer's (2003) [1]. We also dropped the unit vector questions because of high student success rates (80-90% correct). Most of the incorrect responses from questions about unit vectors, magnitude, and direction resulted from poor arithmetic or attentiveness, and did not reflect a lack of student understanding on the topics.

We added to the MC tests a question dealing with the incorrect belief of a dot product having a direction. In interviews, students draw in vectors for the question on dot product magnitude. More on that finding will follow below.

## RESULTS AND DISCUSSION

Early analysis of the multiple choice test indicates the mean overall performance on both tests are about the same, at 38% correct on math and 36% correct on physics. Below we only highlight some preliminary observations (see Figure 3) from the interviews and MC tests pertaining to the questions on: one-dimensional vector subtraction (Fig. 1),

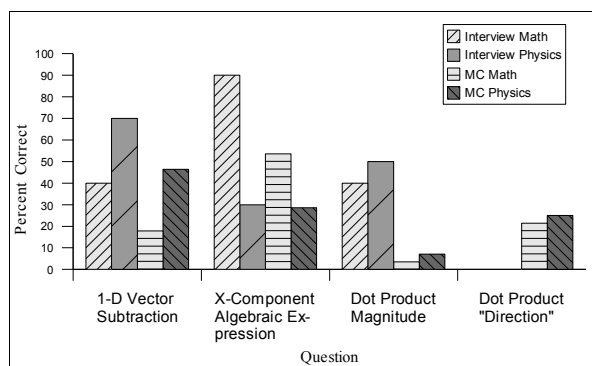
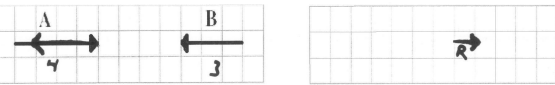
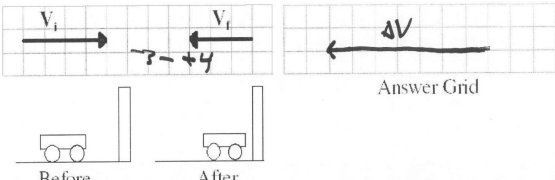


Figure 3. Preliminary interview (N=10) and MC (N=28) data.

$x$ -component algebraic expression (Fig. 2), and dot product.

### One-Dimensional Vector Subtraction

We observed a discrepancy between the number of correct answers on the math and physics tests for the one-dimensional subtraction question (see Fig. 3). Interview results show students using added tools when solving the physics problem than were used for the math problem. Many students answering incorrectly on the math test subtracted the magnitudes of the vectors rather than subtracting the vectors themselves. One student's responses to the math and physics questions are given in Figure 4. Her correct response on the physics question is consistent with that of other students who interpreted the minus sign as a marker of direction and implicitly labeled the directions in the physical system with a coordinate system.

Math Version
<p>2. Consider vectors <b>A</b> and <b>B</b> below. Find <b>R</b>, the difference vector, when <b>A</b> is subtracted from <b>B</b> (i.e. <math>\mathbf{R} = \mathbf{B} - \mathbf{A}</math>).</p>  <p style="text-align: center;">Answer Grid</p> <p><b>Student:</b> (Draws <b>R</b>) The resultant vector is one, four minus three, one basically. (writes 4 and 3)</p>
Physics Version
<p>2. Consider a cart colliding into a wall. Its initial velocity <math>\mathbf{V}_i</math> and the final velocity <math>\mathbf{V}_f</math> are given as depicted below. In the space provided draw and label the change in velocity of the cart, <math>\Delta\mathbf{V}</math>, where <math>\Delta\mathbf{V}</math> is <math>\mathbf{V}_f</math> subtracted from <math>\mathbf{V}_i</math> (i.e. <math>\Delta\mathbf{V} = \mathbf{V}_f - \mathbf{V}_i</math>).</p>  <p style="text-align: center;">Answer Grid</p> <p><b>Student:</b>...which is three minus negative four. Which is...negative three, minus positive four, there we go. Which is negative seven (points to <math>\Delta\mathbf{V}</math>).</p> <p><b>Interviewer:</b> How did you decide which is positive and which is negative?</p> <p><b>Student:</b> I was just going um...positive velocity is in the <math>x</math> direction, kind of before was positive plus it's just <math>x</math> direction (motioning right with pen). I usually go right as always positive. General the way I do things.</p>

**Figure 4.** 1-D vector subtraction scanned interview sketches.

Of the seven students who answered correctly in the physics version, six used this method. More

interestingly, two of the students who still responded incorrectly in the physics version also used the above tools and derived the correct response. However, these two students noticed the inconsistency between their previous incorrect math answer and their current physics answer. Their response to this inconsistency was to justify their previous math answer and disregard their reasoning on the physics version, resulting in answers like the one shown in the math version of Figure 4.

We also observe such large differences in performance between the math and physics one-dimensional vector subtraction question in the MC data (see Fig. 3). We found overall low performance, with 18% correct on math and 46% correct on physics. The most common incorrect answer on both versions is that shown in the math version in Figure 4. Learning between question versions could account for the increase in correct responses on the physics version since students completed the physics version immediately after the math version.

### Algebraic Expression for a Vector Component

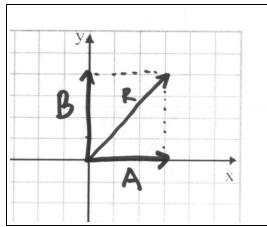
In the algebraic component-magnitude question, students are expected to find an algebraic expression for the  $x$ -component of a vector ( Fig. 2). During interviews, nine students answered correctly on the math version and only three on the physics version (see Fig. 3). We often give students inclined-plane or ramp problems similar to this one, and presume students will take that information and appropriately label the known angle with reference to the vector, and coordinate system at hand. However, this subtle difference in versions proves to be a challenge for students. The majority of incorrect answers on the physics version result from inappropriate re-labeling of the angle: students place theta between the  $x$ -axis and vector **N** instead of the  $y$ -axis and vector **N**.

In the MC data, we find low performance for these questions, with 54 % correct on math and 29% correct on physics (see Fig. 3). The only incorrect choice selected on the physics version was choice “e” (see physics version answers in Fig. 2), indicating a large portion of students leaving our summer-session calculus-based introductory physics class still have difficulty with their relabeling of angles.

### Dot Product

During interviews, students performed poorly on questions about dot product magnitudes on both the math and physics versions (see Fig. 3). Unexpectedly, two students drew in a vector for the dot product.

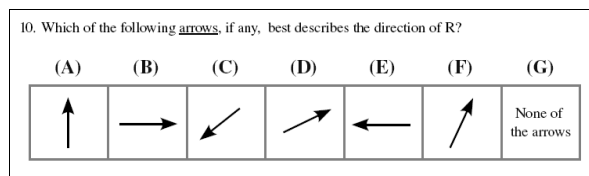
They were using vector addition tools to evaluate a dot product. Figure 5 shows a student using the parallelogram method of vector addition to find  $R$ , which was defined as the dot product of vectors  $A$  and  $B$  in the question. Both students consistently indicated direction of a dot product on the isomorphic physics question dealing with work.



**Figure 5.** Incorrect interview response to the math dot product question. Students were first asked to draw in the vectors  $A = 3i$  and  $B = 4j$ . Then students were asked to find  $R$ , the dot product of vector  $A$  and  $B$ . In the MC questions vectors  $A$  and  $B$  are given.

We observed students performing extremely poorly on the MC dot product magnitude question, with 4% correct on math and 7% correct on physics. When asked to find the dot product of vectors  $A$  and  $B$  above the most common incorrect choice (~50%) was  $3i+4j$ , indicating student use of vector addition tools to evaluate a dot product. The math version may prime students to use vector addition tools through its use of the non-neutral symbol  $R$ , which is typically used to represent a resultant vector. However, the same fraction of students incorrectly answered  $3i+4j$  in the isomorphic physics version in the context of work, suggesting the math notation did not impact student responses.

The majority of students ended the semester by incorrectly applying “direction” to dot products. The MC tests results show students answering correctly only 21% in math version and 25% in the physics version on the dot product “direction” question (see Fig. 3). The MC math version of the dot product “direction” question is shown in Figure 6. The most common incorrect choice in both versions (~75% math and ~65% physics) was the arrow pointing in the direction of the vector sum of vector  $A$  and  $B$  (choice F in Fig. 6).



**Figure 6.** MC math dot product “direction” question.  $R$  is defined in the previous dot product magnitude question as being equal to  $A \cdot B$ , where  $A = 3i$ , and  $B = 4j$ . The correct

answer is  $G$ , *none of the arrows*, because a dot product is a scalar and does not have direction.

## CONCLUSION

Introductory students' overall performance on our isomorphic math and physics MC vector tests was very similar, 36% and 38% correct respectively. However, closer preliminary analysis of the interviews and MC questions reveals differences. Our introductory physics students often use different tools on the math and physics one-dimension graphical vector subtraction questions. In an interview setting students were observed in the math context to subtract vector magnitudes without regard to direction. When answering the physics question, students used the minus sign and an implicit coordinate system to label direction and appropriately subtract the vectors. MC data indicates a greater portion students subtracting vector magnitudes on the math version than on the physics version. During interviews we found student difficulties with relabeling of angles on an inclined-plane problem, specifically when the presented angle is the angle of the incline; MC data support this finding. Finally, we observed student confusion about dot products, including difficulties with “direction.” Specifically, students used vector addition tools to evaluate a dot product in an abstract math context and in the physics context of work.

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