48) had a correct interpretation. More than just this quarter of students may have recognized the constant's physical significance but did not explicitly write it down. The interviews suggest that a students' interpretation of the constant can be facilitated or impeded by their identification of its units. This dynamic will be discussed in greater detail in relation to the Reflection component (below).

Execution of the mathematics: One exam question provided an expression for the charge density of three point charges and asked for $\int \rho(\vec{r})d\tau$. Roughly a quarter of the students (27%, N=15 of 56) made significant mathematical errors related to the delta function while executing this integral (element E1). The most common error (73%, N=11 of 15) amounted to a variation of equating the integral of the delta function with the integral of its vector argument. This difficulty was also implicit in one third (32%, N=27 of 84) of the responses to the CUE.

The second interview set (N=6) targeted the first element in Execution differently by asking students to perform the context-free integrations shown in Fig. 2. Two students stated that the integral in part b) would be equal to x without evaluating this expression at $x = 0$, but none of the six participants had difficulty with the integrals in parts a) or c). This level of success is somewhat surprising given that a quarter of the exam students struggled to execute integrals that, to an expert, are very similar. One explanation may be that the $\delta^3(\vec{r})$ notation used on the exam was harder for students to deal with than the mathematically equivalent $\delta(x)\delta(y)\delta(z)$. Three of six interviewees also evaluated the r integral in part d) as if the delta function was not there (i.e., $\int \delta(r-r')r^2dr = \frac{1}{3}r'^3$), despite correctly executing parts a)-c). Their verbal explanations indicated that the issue was the delta function rather than the spherical integrals. These results again suggest that students' success at common delta function integrals may not transfer to more complex integrals.

Reflection on the result: For the questions used in this study, one of the most powerful tools available for checking and interpreting the various delta function expressions is looking at units (elements R1 and R2). When asked for the units of the given constant (e.g., c in Fig. 1*(a)*), two thirds of the exam students (69%, N=128 of 186) gave correct units. We would also like our students to consider the physical meaning of this unitful constant (element C3), but it was often difficult to assess if they had done so on our exam questions. However, a third of students (32%, N=60 of 186) gave units that were inconsistent with the geometry they identified. This pattern indicates that they either did not have an appropriate physical interpretation of this constant (elements C3) or failed to connect that interpretation to the units (element R2).

The interviews offer additional insight into the connection between the units and physical interpretation of the constant. When prompted to comment on units, 9 of 11 participants explicitly argued (incorrectly) that delta functions were unitless and thus, regardless of the geometry of the charge distribution, the units of the constant must be C/m^3. This argument was often justified by the statement that the delta function was 'just a mathematical thing' and thus did not have units. Four of these students had previously expressed a correct physical argument for the units of the constant. In each case, the student either abandoned their physical interpretation or were unable to reconcile these conflicting ideas. Ultimately, 7 of these 9 students required help from the interviewer to convince themselves of the units of the delta function.

CONCLUDING REMARKS

This paper presents an application of the ACER framework to guide analysis of student difficulties with the Dirac delta function in the context of mathematically expressing charge densities in junior-level electrostatics. We find that our upper-division students have difficulty; (1) activating delta functions as the appropriate mathematical tool when not explicitly prompted, (2) translating a verbal description of a charge distribution into a mathematical formula for volume charge density, (3) transferring their knowledge of how to integrate delta functions to more complex and novel integrals, and (4) determining the units of the delta function in order to reflect on or check expressions for the charge density.

These findings have several implications for teaching and assessing the use of delta functions in electrostatics. Instructors should be aware that the canonical delta functions questions rarely require a student to consider when delta functions are appropriate. Furthermore, constructing a mathematical expression for the charge density is a more challenging task than interpreting that same expression. Additionally, the belief that the delta function is unitless was a surprising prevalent and persistent difficulty that may be exacerbated by presenting the delta function as a purely abstract mathematical construct.

The ACER framework provided an organizing structure for our analysis that helped us identify nodes in students' work where key difficulties appear. It also informed the development of interview protocols that targeted aspects of student problem solving not accessed by traditional exams. The difficulties identified in this paper represent a subset of students' difficulites with the Dirac delta function and may not include issues that might arise from its uses in contexts outside of electrostatics.

This work was funded by the NSF (CCLI Grant DUE-1023028 and GRF under Grant No. DGE 1144083).

REFERENCES

1. D. Meltzer & R. Thornton, *Am. J. Phys.* **80**, 478 (2012).
2. B.R.Wilcox, *et al.*, *Phys. Rev. ST-PER* **9**, 020119 (2013).
3. D. Hammer, *Am. J. Phys.* **68**, S52–S59 (2000).
4. D.J. Griffiths, *Introduction to electrodynamics*, Prentice Hall, 1999, ISBN 9780138053260.
5. S.V.Chasteen, *et al.*, *Phys. Rev. ST-PER* **8**, 020107 (2012).