

Mathematization and the ‘Boas course’

Michael E. Loverude

¹*Department of Physics, California State University Fullerton
Fullerton, CA 98031*

Abstract. Research on the use of mathematics in physics has included empirical and theoretical studies. We consider the implications of these studies on the mathematical methods course offered by many physics departments, often referred to as the ‘Boas course’ after a common textbook. Surveys of students entering such a course suggest that for many students the math in introductory courses consisted primarily of plugging numbers in formulas and execution of algebraic or arithmetic procedures. Data suggests that despite experience with procedures, many students entering math methods do not make sense of mathematical ideas relevant to upper-division physics. As a result, students arrive ill-prepared for physicist math and sensemaking. Models of learning and learning transfer suggest strongly that students will not spontaneously develop these skills by performing procedural exercises. The math methods course presents an ideal opportunity to develop these skills by explicitly practicing them in physics contexts.

I. INTRODUCTION

This work is part of a collaboration to investigate student learning and application of mathematics in the context of upper-division physics courses, particularly the math methods courses offered in many physics departments. Our project seeks to study student conceptual understanding in upper-division physics courses, investigate models of transfer, and to develop instructional interventions to assist student learning.

While PER has primarily focused on introductory-level courses, there are increasing efforts to expand into the upper division [1]. The core sophomore- and junior-level courses taken by most physics majors have begun to receive the attention of researchers and curriculum developers. One key course that remains under-researched (with a few exceptions [2, 3]) is the intermediate mathematical methods course taught by most departments. This course, which we describe as *Math Methods*, is often informally known as the ‘Boas course,’ from a standard text [4]. While there are multiple models for such a course, the most common is one that comes after introductory-level physics and calculus courses and before upper-level physics theory courses.

For this paper, we seek to summarize previous work, both empirical and theoretical, and to make and support additional claims based on our own research. These claims will together be used to advance an argument that many students entering upper-division physics often are experienced with procedures but have little experience with mathematization and sensemaking, and that a blended math-physics environment is needed. By mathematization, we follow Brahmia et al, who describe the use of mathematics to quantify and conceptualize the physical world. [5] We do not include within this term procedural fluency, but rather the ability to make sense of physical

systems using mathematics. The ‘Boas course’ is a key opportunity to develop the quantitative reasoning skills that physicists value, but without a theory- and data-informed understanding of the students and how and what they have learned, the opportunity will be missed.

II. BACKGROUND AND CONTEXT

A. Background and relevant prior work

A growing body of work in PER has examined student use of mathematics in physics, with particular emphasis on the upper division. Many studies have focused on student difficulties. For example, Pepper and colleagues [6] describe student difficulties with upper-level E&M, noting in particular that students ‘struggle to combine mathematical calculation and physics ideas...and to access appropriate mathematical tools.’

Several studies have applied a resources framework [7] to describe students’ use of math in physics problems. Sayre and Wittmann used a resource framework and described ‘plasticity’ of resources in the process of choice of coordinate system. [8] Vega described student resources in unit vectors in Math Methods [9].

Beyond particular contexts, several models have been proposed to describe student use of mathematics in physics. Redish has proposed a framework to describe student usage of mathematics in science course [10], describing stages of *modeling, processing, interpreting, and evaluating*. For upper-division physics courses, Wilcox et al. have proposed the ACER framework ‘to guide and structure investigations of students’ difficulties with the sophisticated mathematical tools used in their physics classes.’ [11] In this framework, students must *activate* the appropriate mathematical tool in addition to *constructing* a model, *executing* the mathematics, and then *reflecting* on results.

Uhden et al [12] proposed a model in which the degree of mathematization is represented vertically, with more mathematical descriptions higher on a diagram. Upward movement represents the process of mathematizing, by which they mean constructing a mathematical representation of a system. Downward arrows represent ‘interpreting the physical meaning of mathematical expressions.’ In the model, exclusively mathematical operations are represented in a different space; students leave the mathematization space to ‘just do math’ in this space and then return to the same location after completing the calculation.

In each of these models, successfully executing the mathematical procedure in question is only one element of success. Each explicitly notes the importance of constructing a mathematical model from a physical system. The models above also at least imply that physics and mathematics are distinct. Bing and Redish instead describe conceptual blending [13], in which students operate in a conceptual space that shares elements of mathematical and physical reasoning.

B. Context for research

The empirical portion of this work has taken place in the context of two courses taught at a large public comprehensive university serving a diverse student population. Both are theoretically-oriented core courses that are required for physics majors; for most students, these are the first upper-division physics courses. Thermal Physics is an intermediate-level course in thermal and statistical physics. The other course, Math Methods, uses the text by Boas [4] and covers a fairly standard list of topics. Each course meets for two 75-minute blocks per week. The courses have as prerequisites three semesters of calculus, and most students have completed two or three semesters of introductory physics. The author has taught each course multiple times. Through the period of this study, course enrollments were between 12 and 24.

III. CENTRAL CLAIMS AND SUPPORT

We base our argument on four claims. For this short report, we offer only brief descriptions of the support for each claim, but additional data and citations are available.

A. Students encounter less calculus than expected in calculus-based physics

Students in the Thermal Physics course ($N = 21$) were asked to comment in an online essay on introductory level and upper division courses, and specifically comment on mathematics and calculus in intro courses. (About half took at least one of these courses at other institutions.) The results suggest that while there is variation in the mathematical expectations of the lower division courses,

that the typical physics major in this sample is not encountering much calculus in introductory physics. One student described the math as ‘extremely relaxed and somewhat inconsistent.’ Another described the course sequence as ‘just barely calculus based.’ Every verb in the sample was identified; verbs included more than once included: plug, doing/done, crunch, memorize, solve, derive, and shown (as in ‘we were shown’). The responses suggest that the mathematics in the introductory course focused primarily on computing answers to end-of-chapter textbook problems. Although physics majors take a calculus-based introductory sequence, their responses suggest that most of the calculus was done by the instructor in lecture rather than by students:

The calculus that we used was mentioned once, and then the students were expected to just memorize the equations of motion instead of deriving them ...

...there were many calculus based derivations that we weren't in any way required to know or understand. We only just used their results to do plug and chug style problems.

Another student noted,

We mostly used the results of some equations to solve problems, so pretty much algebra.

Despite the reputation of introductory physics, it seems that many students don't actually do a lot of math beyond arithmetic and algebra. (None of the students mentioned trigonometry, and only one mentioned vectors, though that may simply reflect that the prompt mentioned math and calculus and not these other topics.)

B. Exposure to mathematical procedures does not mean that students make sense of the math

We previously reported that many students in the Math Methods course had difficulty in responding to tasks involving mathematical ideas that many instructors would assume had been learned in prior courses [14]. There are cases in which students have not encountered ideas in prior course work, but in most cases students exhibit familiarity with mathematical procedures but nevertheless struggle to make sense of the math.

For example, describing waves and oscillations using complex numbers is essential in upper-division physics. After experience suggested that students found this difficult, we probed their understanding of the connection between complex numbers and oscillations. In the Math Methods course, students were asked to sketch on a blank graph the real part of the function $f(t) = Ae^{i\omega t}$ as a function of time and identify any relevant points. The expectation was that students would sketch a cosine function with amplitude

A (i.e., value A at $t=0$). This problem was posed on an ungraded quiz in four sections of the course ($N = 49$), after reading and preliminary lecture on the topic but before a tutorial exercise on complex numbers and oscillations. Students informally professed familiarity with complex numbers and Euler's equation from previous courses and were generally successful with procedural tasks using complex numbers (less so with polar form).

Student responses were categorized based on the following features: oscillatory or not, value at $t=0$, constant or varying amplitude. Ten percent of responses could not be classified with this scheme and required an additional category; some included an arrow (possibly in the complex plane) rather than a sketch of the function. About 40% of responses were categorized as correct or nearly correct, including an oscillatory function with or without the correct value at $t=0$. Over 40% of responses were categorized as graphs not including oscillation; most represented exponential *growth*. Sadaghiani reported similar confusion between e^{kx} and e^{ikx} [15].

This is only a single example, of course, but it suggests that experience with mathematical procedures does not necessarily mean students make sense of the meaning of the procedures. Most of the sections on complex numbers in the Boas text include only procedural exercises. The applications, including oscillations are separated into a later section with only a handful of examples or problems.

C. Many students enter the upper-division courses without much experience in quantitative sensemaking

In recent work [16], we have described a set of quantitative reasoning skills that are valued by physicists but not often explicitly taught or assessed. For the purpose of this paper, we will refer to questions designed to assess several quantitative reasoning skills. The set of these skills is not intended to be complete, but we have identified several that appear to be relevant as starting points:

- Using dimensional / unit analysis
- Testing expressions with limiting cases
- Using approximations, e.g., with Taylor series
- Sketching functions
- Identifying errors in solutions
- Predicting the effects of problem changes on the resulting solution

Our data suggest that many students entering the Math Methods course do not successfully reason quantitatively even when explicitly prompted to do so. For example, we described student responses to an ungraded quiz posed on the first day of the Math Methods course, in which students were shown a diagram with an Atwood's machine and given three proposed expressions for the acceleration of one of the blocks. Students were asked to evaluate whether each of the possible expressions could be correct without solving the problem. For example, students might check the units, considering limiting cases (if one block and the

pulley are massless, the other block should be in free fall). After collecting written data, we also posed the question in a think-aloud interview setting.

The responses given by students suggest that they do not recognize the nature of these tasks and that they have little experience in such reasoning. In written responses, essentially none of the students used limiting cases and very few successfully made sense of the equations. In the interviews, students who had completed a traditional Math Methods course similarly struggled to decide whether the expressions were correct. When prompted to do so, students were able to use limiting case arguments with varying degrees of success, but none did so spontaneously. Although physicists value these skills and models like ACER describe them as essential, traditional instruction, focused almost entirely on procedures, does not lead students to develop other quantitative reasoning skills.

D. Models of learning transfer suggest strongly that students need to learn math in context

The process of applying mathematical knowledge in physics calls back to the classic cognitive question of transfer. Standard models of transfer presume decontextualized and portable knowledge. Differences between the context in which an idea, e.g., integration, was originally learned and the physics problem to which students apply integration are interpreted as "surface features." The math requirements for many courses reflect such an understanding of transfer. The expectation is that students learn to integrate in math, and then when students encounter appropriate physics problems, they employ the well-understood tool of integration.

This classical model of transfer has been challenged by many researchers. First, more attention is being given to the social environments in which learning (and later transfer) take place. Transfer across mathematics and physics is likely to be affected by differences in the culture, language, and representations of the disciplines.

Second, from a cognitive perspective, Wagner's transfer-in-pieces approach [17] suggests that knowledge of at least some mathematical concepts is highly context-sensitive, and that mathematical ideas perceived as experts as being "the same" require different supporting knowledge to be used in different contexts. Contextual differences cannot be dismissed as mere surface features, but as affordances for using different knowledge resources in order to see "the same thing" in two different contexts. Integration problems look different in physics and math, as well as in different areas of physics. Transfer is not simply a matter of employing a well-understood tool, but rather of broadening one's understanding of the meaning of integration to include the context in which it is employed.

These models suggest that decontextualized procedural examples will not help students who struggle to make sense of mathematics in physics contexts. They further suggest

that instructors should make explicit efforts to put students in situations in which they must broaden the scope of their understanding of a concept to include new contexts.

IV. IMPLICATIONS FOR THE ‘BOAS’ COURSE

The Boas course and textbook embodies several tensions. Is the course simply a math course taught by a physicist, or a physics course in which math is at the forefront? Is the purpose of the course to learn new math (or relearn previously covered math) or to do physics problems that involve representative mathematics concepts? Indeed, some physics departments do not offer Math Methods, instead simply requiring that students take six semesters of math rather than four.

This paper makes the claim that the choices made by instructors and textbook authors should reflect the four claims from Section III. Indeed, these four claims suggest that the Math Methods course offered after introductory physics but before upper-division physics does fulfill a need. Many students do not get much experience in using calculus in physics contexts or in making sense of math in their introductory courses. Prior courses have likely focused on procedural fluency and computing with formulas, not on sense making. Instructors cannot assume that physics majors entering the upper division have developed sense making tools that physicists value.

The Math Methods course should therefore be different from any other math or physics course, and should in fact reflect a truly blended space. It is not a math course taught by mathematicians, but one in which the primary focus is

math as done by physicists. It is unlike most other physics courses, though, because the physics topics covered do not tell a coherent physics story; rather, the physics topics are chosen to illustrate how physicists use certain mathematical principles and techniques. Simply teaching procedures, as if it were a math course, is a lost opportunity.

Redish and Kuo [10] write that students “need to learn a component of physics expertise not present in math class—tying those formal mathematical tools to physical meaning.... We as physics instructors must explicitly foster these components of expert physics practice to help students succeed in using math in physics.” The Math Methods course is perfectly positioned in the curriculum to fill this role for physics majors. The problems in the course text that are merely mathematical exercises do not serve this need, however; they do not include physical contexts, units, or parameters. Additional work should seek to understand and promote this nonprocedural learning.

ACKNOWLEDGEMENTS

This work is part of a collaboration with John Thompson (University of Maine and Warren Christensen (North Dakota State University). The author acknowledges student researchers Marlene Vega (CSUF) and Mikayla Mays (CSUF). This work is supported by the NSF through grant PHYS-1405616; opinions, findings, and conclusions are those of the author and do not necessarily reflect the views of the NSF.

References

- [1] M. E. Loverude and B. S. Ambrose, “Editorial: Focused Collection: PER in Upper Division Physics Courses,” *Physical Review Special Topics in Physics Education Research* 11, 020002 (2015).
- [2] See [arXiv:1207.0987v2](https://arxiv.org/abs/1207.0987v2) [physics.ed-ph] SEI: Science Education Initiative at University of Colorado Boulder, <http://www.colorado.edu/sei/departments/physics.htm>
- [3] E. F. Redish, “Problem solving and the use of math in physics courses.” In Conference World View on Physics Education in 2005, Delhi, August 21–26, 2005..
- [4] M. Boas, *Mathematical Methods in the Physical Sciences*, Wiley (2006).
- [5] S. Brahmia, A. Boudreaux, and S. Kanim, “Obstacles to Mathematization in Introductory Physics,” submitted to *Physical Review Physics Education Research*, <https://arxiv.org/pdf/1601.01235.pdf>
- [6] R. E. Pepper, S. V. Chasteen, S. J. Pollock, and K. K. Perkins, “Observations on student difficulties with mathematics in upper-division electricity and magnetism,” *Physical Review Special Topics Physics Education Research* 8, 010111 (2012)
- [7] D. Hammer, “Student resources for learning introductory physics,” *American Journal of Physics* 68, S52–S59 (2000).
- [8] E. C. Sayre and M. C. Wittmann, “Plasticity of intermediate mechanics students’ coordinate system choice,” *Phys. Rev. Spec. Topics: PER*, 4, 020105 (2008).
- [9] M. E. Vega, W. Christensen, B. Farlow, G. Passante, and M. Loverude, “Student understanding of unit vectors and coordinate systems beyond cartesian coordinates in upper division physics course,” PERC Proceedings 2016
- [10] E. Redish & E. Kuo, “Language of Physics, Language of Math: Disciplinary Culture and Dynamic Epistemology,” *Science and Education* 24, 561-590 (2015).
- [11] B. Wilcox et al, *Physical Review Special Topics – Physics Education Research* 9, 0201191 (2013).
- [12] O. Uhden et al., “Modelling mathematical reasoning in physics education,” *Science & Education* 21 (4), 485-506 (2012).
- [13] T. J. Bing and E. F. Redish, “Epistemic complexity and the journeyman-expert transition,” *Physical Review Special Topics – Physics Education Research* 8, 010105 (2012),
- [14] M. E. Loverude, “Surprisingly, there is an actual physical application,” Student understanding in Math Methods, 2013 PERC Proceedings [Portland, OR, July 17-18, 2013], edited by P. V. Engelhardt, A. D. Churukian, and D. L. Jones.
- [15] H. R. Sadaghiani, “Conceptual and mathematical barriers to students learning quantum mechanics,” Ph.D. Dissertation, the Ohio State University (2005).
- [16] M. E. Loverude, “Beyond Procedures: Quantitative Reasoning in Upper-Division Math Methods in Physics,” RUME conference proceedings
- [17] J. F. Wagner, “Transfer in pieces,” *Cognition and Instruction* 24, 1–71 (2006).