Using a clicker question sequence to teach time-development in quantum mechanics

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Research-validated clicker questions as instructional tools for formative assessment are relatively easy to implement and can provide effective scaffolding when developed and implemented in a sequence. We present findings from the implementation of a research-validated Clicker Question Sequence (CQS) on student understanding of the time-development of two-state quantum systems. This study was conducted in an advanced undergraduate quantum mechanics course. The effectiveness of the CQS was determined by evaluating students' performance after traditional lecture-based instruction and comparing it to their performance after engaging with the CQS.

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I. INTRODUCTION

The time-evolution of a quantum state is an important concept in quantum mechanics. Many fields of active research, including quantum computing, must contend with the dynamical behavior of quantum systems. Since it draws on prerequisite knowledge of quantum states and the Hamiltonian of the system, the concept can be challenging for students to grasp. At the advanced undergraduate level, time-evolution of a quantum state is introduced with a time-independent Hamiltonian $\hat{H}$. The state as a function of time t is then the solution to the time-dependent Schrödinger equation, i.e., $i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$, and is equivalent to applying the operator $e^{-\frac{i\hat{H}t}{\hbar}}$ to the initial state.

Because the Hamiltonian governs the time-development of the state, the eigenstates of the Hamiltonian, i.e., the energy eigenstates or "stationary states," are special in that they simply acquire an overall time-dependent phase factor. For instance, given a Hamiltonian $\hat{H} = C\hat{S}_z$ for a two-state system with a dimensionally-appropriate constant C, an initial state expressed in the energy eigenbasis is $|\chi(t=0)\rangle = a|z\rangle + b|-z\rangle$. The state at time t is $|\chi(t)\rangle = ae^{-\frac{iCt}{2}}|z\rangle + be^{\frac{iCt}{2}}|-z\rangle$, since $\hat{H}|z\rangle = \frac{C\hbar}{2}|z\rangle$ and $\hat{H}|-z\rangle = -\frac{C\hbar}{2}|-z\rangle$. Here all notations are standard. If, however, the initial state is expressed in some other basis, one can obtain the state at time t by first re-expressing the initial state as a superposition of energy eigenstates before introducing the time-dependent phase factors to each term.

To become proficient at determining the state at time t given an initial state in some basis, students must be adept at several different tasks. These include being able to recognize whether the given initial state is an eigenstate of the Hamiltonian; working in the energy eigenbasis and converting to this basis if the initial state is given in any other basis; and correctly applying the time-evolution operator. Students also must recognize that different energy eigenstates generally correspond to different eigenvalues. The convergence of all these challenges, as well as possible unfamiliarity with the meaning of the complex exponential itself, can place significant demands on students' cognitive resources. This may also obfuscate other consequences of the Hamiltonian playing a central role, e.g., the expectation value of any observable (that does not have any explicit time-dependence) does not depend on time in a stationary state.

Prior research suggests that students in quantum mechanics courses often struggle with many common difficulties, including with issues related to time-development of a quantum state, but research-validated learning tools can effectively help students develop a robust knowledge structure [1–37]. Quantum Interactive Learning Tutorials (QuILTs) have been developed, validated and implemented on many topics in quantum mechanics, with encouraging results [19,38,39]. Similarly, clicker questions, first popularized by Mazur using his *Peer Instruction* method, are conceptual multiple-choice questions presented to a class for students to answer anonymously, individually first and then again after discussion with peers, and with immediate feedback. They have proven effective and are relatively easy to incorporate into a typical course, without the need to greatly restructure classroom activity or assignments [40,41]. When presented in sequences of validated questions that build on one another, they can systematically help students with a particular theme that they may be struggling with. Previously, such Clicker Question Sequences (CQS) have been developed, validated and implemented on several key topics in quantum mechanics [42–46]. Here we discuss the development, validation and implementation of a CQS focused on helping students learn time-evolution of two-state quantum systems.

II. METHODOLOGY

The CQS targets upper-level students in junior-/senior-level quantum mechanics courses. The data presented here is from implementation in a mandatory junior-/senior-level course at a large research university, with sample size N = 29. To develop and validate this CQS, we took advantage of the learning objectives and goals of the QuILT on this topic that had previously been developed [38,39,45]. Taking inspiration from the validated pre- and post-tests intended for use with that QuILT, we made adjustments to questions to specifically address the time-development of two-state systems. Additional inspiration came from questions from other sequences, including those focused on the time-development in the context of Larmor precession.

Additionally, we took advantage of much of the cognitive task analysis both from the expert and student perspectives (based upon interviews) and the scaffolding that had been incorporated in the aforementioned QuILT. We focused on condensing this material, to ensure that the CQS can be administered in class. To be strategic with regard to the available class time, we prioritized basic conceptual knowledge and specific consequences that students often find difficult, provided checkpoints at which instructors should discuss some broader themes related to the previous questions, and avoided burdensome calculations.

After we conceptualized the most important features of time-development of quantum states that students should know, we drafted questions and discussed among ourselves many times to minimize unintended interpretations. We standardized terminologies and sentence constructions while simplifying them as much as possible to avoid causing cognitive overload for students. We also paid attention to the answer choices for each question. In some instances, after discussion amongst researchers, we revised the questions to make sure that students understood them unambiguously.

We aimed to address common stumbling blocks and emphasized key features that students may have missed in a typical lecture. The 13 questions in the CQS focused on four learning goals on the following topics: identifying the basic properties of the energy eigenstates or stationary states (CQS 1.X, 2 questions), transforming from an initial state to its time-evolved state (CQS 2.X, 5 questions), expressing a state in the energy eigenbasis before applying the time-evolution operator (CQS 3.X, 4 questions), and calculating the time-dependence of various observables' expectation values (CQS 4.X, 2 questions). We designed several questions specifically to address certain student difficulties that have previously been found [12,13,20]. Selected CQS questions, referenced in later sections, are reproduced below (answers in boldface, all notations being standard and familiar to students):

CQS 1.2 Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$. Which of the following initial states $|\chi(t=0)\rangle$ is a stationary state?

I. $|\chi(0)\rangle = |z\rangle$

II. $|\chi(0)\rangle = |x\rangle$

III. $|\chi(0)\rangle = a|z\rangle + b|-z\rangle$, because $|z\rangle$ and $|-z\rangle$ are both stationary states.

A. All of the above **B. I only** C. I and II only
D. I and III only E. None of the above

CQS 2.2 Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$. Choose all the correct statements about a system in the state $|\chi(0)\rangle$.

I. $|\chi(t)\rangle = e^{\frac{-iE_n t}{\hbar}}|\chi(0)\rangle$, where E_n is an eigenvalue of $\hat{H}$.

II. $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}}|\chi(0)\rangle$, where $\hat{H}$ is the Hamiltonian of the system.

III. Each measurement of a generic observable Q will return the same result, regardless of the time when the measurement is performed.

A. II only B. III only C. I and II only
D. II and III only E. None of the above

CQS 3.1 Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$. Choose all the correct statements about a system in the state $|\chi(0)\rangle$.

I. $|\chi(0)\rangle = (|z\rangle\langle z| + |-z\rangle\langle -z|)|\chi(0)\rangle$, where $|z\rangle\langle z|+|-z\rangle\langle -z| = \hat{I}$

II. $|\chi(0)\rangle = C_1|z\rangle + C_2|-z\rangle$, where $C_1 = \langle z|\chi(0)\rangle$ and $C_2 = \langle -z|\chi(0)\rangle$ are both coefficients

III. $|\chi(t)\rangle = e^{\frac{-iE_+ t}{\hbar}}C_1|+z\rangle + e^{\frac{-iE_- t}{\hbar}}C_2|-z\rangle$

A. I only B. II only C. I and II only
D. II and III only **E. All of the above**

CQS 3.2 Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$. Choose all of the following that are correct about a system in the state $|\chi(0)\rangle = a|x\rangle + b|-x\rangle$.

I. $|\chi(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}}|\chi(0)\rangle$

II. $|\chi(t)\rangle = ae^{\frac{-iE_+ t}{\hbar}}|x\rangle + be^{\frac{-iE_- t}{\hbar}}|-x\rangle$

III. To find $|\chi(t)\rangle$, we can write $|\chi(0)\rangle$ as a linear superposition of energy eigenstates, and then attach a time-dependent phase factor with the appropriate energy to each term.

A. I only B. III only C. I and II only
D. I and III only E. All of the above

CQS 3.3 Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$. Choose the correct expression for the time evolved state $|\chi(t)\rangle$ given an initial state $|\chi(0)\rangle = a|x\rangle + b|-x\rangle$.

A. $|\chi(t)\rangle = ae^{\frac{-iE_+ t}{\hbar}}|x\rangle + be^{\frac{-iE_- t}{\hbar}}|-x\rangle$

B. $|\chi(t)\rangle = \frac{a}{\sqrt{2}}e^{\frac{-iE_+ t}{\hbar}}|z\rangle + \frac{b}{\sqrt{2}}e^{\frac{-iE_- t}{\hbar}}|-z\rangle$

C. $|\chi(t)\rangle = \frac{a+b}{\sqrt{2}}e^{\frac{-iE_+ t}{\hbar}}|x\rangle + \frac{a-b}{\sqrt{2}}e^{\frac{-iE_- t}{\hbar}}|-x\rangle$

D. $\boldsymbol{|\chi(t)\rangle = \frac{a+b}{\sqrt{2}}e^{\frac{-iE_+ t}{\hbar}}|z\rangle + \frac{a-b}{\sqrt{2}}e^{\frac{-iE_- t}{\hbar}}|-z\rangle}$

E. None of the above

CQS 3.4 Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_x$. Choose all of the following that are correct about the time development of the state $|\chi(0)\rangle = \frac{1}{\sqrt{2}}|z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$.

I. $|\chi(t)\rangle = \frac{1}{\sqrt{2}}e^{\frac{-iE_+ t}{\hbar}}|z\rangle + \frac{1}{\sqrt{2}}e^{\frac{-iE_- t}{\hbar}}|-z\rangle$

II. $|\chi(t)\rangle = \frac{1}{\sqrt{2}}e^{\frac{-iE_+ t}{\hbar}}|x\rangle + \frac{1}{\sqrt{2}}e^{\frac{-iE_- t}{\hbar}}|-x\rangle$

III. $|\chi(t)\rangle = e^{\frac{-iE_+ t}{\hbar}}|x\rangle$

A. I only B. II only **C. III only**
D. I and III only E. None of the above

CQS 4.2 Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$. If the system is in the state

$$|\chi(0)\rangle = |z\rangle,$$

which of the following expectation values are time-independent?

I. Energy II. S_z III. S_x

A. None of the above B. I only
C. I and II only D. II and III only
E. All of the above

Since the entire course was remote due to the COVID-19 pandemic, the CQS was administered during the online lectures as a Zoom poll while the instructor displayed the questions via the "Share Screen" function. The instructor allowed several minutes for students to vote before revealing the results, and some had the opportunity to explain their responses, before systematically discussing the different options. When a majority of students selected an option that involved alternative conceptions, the instructor would give a hint and ask students to vote again, and ask for volunteers to explain the reasoning behind their choices. In a typical classroom setting, students would have had easy access to one another to discuss their thinking in small groups, but this proved less feasible in the online instructional setting, where students or the instructor predominantly spoke to the whole class.

To determine the effectiveness of the CQS in helping students overcome these common difficulties, we developed and validated pre- and post-tests that had both questions taken directly from the CQS and other questions on topics covered in the CQS. The post-tests were a slightly modified version of the pre-tests, with some changes (e.g., eigenstates of $\hat{S}_x$ being replaced by eigenstates of $\hat{S}_y$) but otherwise remaining conceptually similar. Students were given the pre-test immediately following traditional lecture-based instruction on the topic. After administration of the CQS, which took place over the course of three lecture sessions, students were given the post-test. For both, they were given a 25-minute period at the end of the class session. Two researchers graded the pre-test and post-test, and after discussion converged on a rubric on which the inter-rater reliability was greater than 95%. Questions 3 and 4 were scored with 2 points split between answer and reasoning, and the remainder were all-or-nothing. The pre- and post-test questions are reproduced below:

Q1. (See CQS 2.2)

Q2. Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$, where C is an appropriate constant. Choose all of the following that are stationary states.

I. $\frac{1}{\sqrt{2}}|z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$
II. $a|z\rangle + b|-z\rangle$
III. $a|x\rangle + b|-x\rangle$, where $a \neq b$

a. All of the above　　b. I only
c. II only　　d. I and II only
e. None of the above

Q3. Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_x$ (note: <u>*not*</u> $\hat{H} = C\hat{S}_z$), where C is an appropriate constant. For a system in the state $|\chi(0)\rangle = \frac{1}{\sqrt{5}}|x\rangle + \frac{2}{\sqrt{5}}|-x\rangle$, what is $|\chi(t)\rangle$? Show your work.

Q4. Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_x$, where C is an appropriate constant. For a system in the state $|\chi(0)\rangle = \frac{1}{\sqrt{5}}|z\rangle + \frac{2}{\sqrt{5}}|-z\rangle$, what is $|\chi(t)\rangle$? Show your work.

Q5. Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$, where C is an appropriate constant.
For a system in the state $a|z\rangle + b|-z\rangle$, which of the following expectation values are time-independent?

I. Energy　　II. S_y　　III. S_z

a. None of the above　　b. I only
c. I and III only　　d. II and III only
e. All of the above

Q6. Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_z$, where C is an appropriate constant.
For a system in the state $|z\rangle$, which of the following expectation values are time-independent?

I. Energy　　II. S_y　　III. S_z

a. None of the above　　b. I only
c. I and III only　　d. II and III only
e. All of the above

III. IN-CLASS IMPLEMENTATION RESULTS

In addition to examining the improvement from the pre-test to the post-test, we analyzed student performance on the clicker questions, including the attractiveness of the distractors. The pre-test and post-test results, as well as normalized gain [2] and effect sizes [47], are listed in Table I. The effect sizes for the six questions ranged from 0.45 to over 1, indicating conventionally medium to large effects.

TABLE I. Comparison of pre- and post-test scores, along with normalized gain (g) and effect size as measured by Cohen's *d* (d), for students who engaged with the CQS (N = 29).

Q#	Pre-test μ	Post-test μ	g	d
Q1	28%	72%	0.62	0.99
Q2	24%	72%	0.64	1.08
Q3	67%	83%	0.47	0.45
Q4	40%	71%	0.51	0.72
Q5	24%	69%	0.59	0.99
Q6	24%	59%	0.45	0.73

Many of the common student difficulties were successfully addressed to varying degrees after students engaged with CQS, as follows.

A. Difficulties that decreased significantly after CQS

1. Difficulties with general eigenstates vs. stationary states

The highest normalized gains were seen in the questions that probed students' knowledge of stationary states (Q1 and Q2). The energy eigenstates are stationary states, but students often remember "eigenstates" as stationary states without having recognized the importance of the "energy eigenstates" aspect. These difficulties could be exacerbated if students are shaky on the prerequisite linear algebra, without a clear grasp of what eigenstates and operators mathematically or conceptually are. Moreover, even if students are proficient with the linear algebra in the context of a math course, transferring that knowledge to the context of a quantum mechanics course can still be very challenging. As illustrated by CQS question 2.2, which also appeared as Q1 on the pre-test and post-test, students appeared to understand the distinction between generic eigenstates and energy eigenstates after the CQS. As seen in Table I, 28% of students answered this question correctly on the pre-test; 40% answered correctly during the CQS (not shown); and 72% answered correctly on the post-test, indicating substantial improvement. Additionally, on Q2, students also better recognized that a superposition of stationary states is not a stationary state, with a normalized gain of 0.64. With similar normalized gains and effect sizes, students also learned in Q5 that only the expectation value of energy does not vary with time in a non-stationary state.

2. *Replacing the operator $\hat{H}$ with one eigenvalue E_n*

For CQS questions 2.2-2.5, students chose with substantial frequency an answer option that resulted in a single phase term involving energy instead of a sum of terms (distractor choice I in question 2.2 invokes this idea). This is at least partially due to students not being entirely comfortable with the notation or not understanding the role of the Hamiltonian. After CQS instruction, most students, when asked in the free-response Q3 on the post-test, correctly multiplied each energy eigenstate by a separate phase factor, with a normalized gain of 0.47 (see Table I).

3. *Difficulties with change of basis*

The CQS question 3.4, which asked students to change from the *z*-basis to the *x*-basis, had considerably lower performance than the preceding question 3.3, which had asked the reverse. This is at least partly due to students having less experience with the former transformation, as the latter is the predominant example used to introduce the idea of changing basis. The symmetry between the two cases may be obvious to more experienced problem solvers, but students needed the opportunity to reason through the basis change. Once students learned the importance of working in the appropriate basis, addressed in CQS questions 3.1-3.4, more students correctly answered the corresponding question (Q4) on the post-test, with normalized gain 0.51. We note that the performance on Q4, with an average of 71%, is a bit lower than that on Q3, 83%, as shown in Table I. With the exception of the latter expressing the given state in the correct basis, the two questions were identical. The lower performance on Q4 is likely due to forgetting the basis change, or making a mistake in the process.

B. Difficulties that were less successfully addressed

Improvement on Q6 on the post-test was the weakest, as seen in Table I. The question appeared with small changes on the pre-test and post-test, and as CQS question 4.2. Although there was some improvement from the pre-test to the CQS question, evidence from the post-test suggests very little further improvement. Students must recognize that invariant probabilities of measurement outcomes imply static expectation values, though it appears that more scaffolding is needed to help students learn this concept. Moving forward, we would specifically include suggestions to encourage students to think of an expectation value as an average of a large number of measurements made on identically prepared systems. Students could also benefit from an additional discussion of Ehrenfest's theorem, giving them more tools with which to process these ideas [29].

C. Several examples of class discussion

A particular advantage of the CQS is that it provides opportunity for rich class discussions that can deepen student understanding. Following are examples of such discussion.

Question 1.2 addressed the common difficulty that any superposition of stationary states is itself a stationary state. Initially the correct answer was not even the most popular response. Without immediately giving a full explanation, the instructor noted that, since any state can be written as a superposition of stationary states, selecting this option would imply that every possible state is a stationary state. When the class was allowed to vote a second time, nearly 50% chose the correct answer.

On question 3.2, two students volunteered to explain how the time evolution of the state could not be simplified (expressed without the Hamiltonian operator) by remaining in the $\{|x\rangle, |-x\rangle\}$ basis, and could thus rule out option II.

Question 4.2 asks about the time-dependence of the expectation value of an observable in a stationary state. Despite the instructor's hints, the distribution of answers remained nearly identical both times the polling was opened to students. While the students may not have been able to sufficiently parse the hints individually, it is likely that performance would have improved in a typical classroom setting, if students were given an opportunity to discuss the meaning of the hints and their consequences in small groups.

Opportunities to hold an overall class discussion about salient concepts such as these after students have voted are very important, but ensuring that instructors hold such discussions when they are recommended can be a challenge especially because time is limited. We will continue to investigate ways to encourage such discussions via check points between CQS questions, even in instances when the instructor may opt not to follow our suggestions verbatim.

IV. SUMMARY

Clicker question sequences can be effective when implemented alongside traditional classroom lectures. We developed, validated, and found encouraging results from implementation of a CQS on the topic of time-development in two-state systems. Post-test scores improved for every question following the administration of the CQS, with mostly uniform normalized gains of around 0.60 in the multiple-choice questions, and high performance in the generative questions that asked students to correctly apply the time-development concepts in open-ended questions. Effect sizes throughout are conventionally large to medium: most were over 0.70. Students' performance was weakest on the questions on expectation values, but we believe that this can be improved through a more robust classroom discussion and more focus on this topic in the CQS itself.

We emphasize that this study was conducted in a remote learning context, and that these results may not transfer exactly to traditional classroom instruction contexts. We will investigate this further in the future.

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