

Development, validation and online and in-person implementation of clicker question sequence on change of basis

Peter Hu, Yangqiuting Li, and Chandralekha Singh

Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260

Research-validated multiple-choice questions comprise an easy-to-implement instructional tool for scaffolding student learning and providing formative assessment of students' knowledge. We present findings from the implementation of a research-validated multiple-choice question sequence on concepts relating to and various methods of changing basis for two-state systems. This study was conducted in an advanced undergraduate quantum mechanics course, in both online and in-person learning environments, across three years. Student learning was assessed after traditional lecture-based instruction in relevant concepts, and students' performance was compared with that on a similar assessment given after engaging with the multiple-choice question sequence. We analyze, compare, and discuss the trends observed in the three implementations.

I. INTRODUCTION

Two-state systems are often used to illustrate many rich phenomena in quantum mechanics (QM), due to their relative simplicity compared to higher dimensional Hilbert spaces. Furthermore, since we are in the midst of the second quantum revolution [1,2], they are critical to the field of quantum information science to describe the behavior of qubits, the smallest unit in which quantum information is stored and processed. Yet because of unfamiliarity with the quantum formalism, even advanced undergraduate students can often struggle with concepts such as basis and changing the basis of two-state quantum systems and knowledge relevant to these concepts. Here, we present research regarding instruction on these concepts, namely outer products and changing between the z -basis and x -basis, two common bases in two-state systems that use the eigenstates of the σ_z and σ_x Pauli matrices, respectively. Outer products are crucial in understanding and being able to carry out a basis change by applying the identity operator expressed in the appropriate basis. The importance of basis and change of basis in two-state systems has been emphasized in prior work [3,4].

Prior research suggests that students in QM courses often struggle with many common difficulties [5–21], including change of basis. For such difficulties, research-validated learning tools can effectively help students develop a robust knowledge structure [22–27]. For example, tools known as Quantum Interactive Learning Tutorials (QuILTs) have been developed with encouraging results on many topics in QM [28–30]. Other commonly used learning tools in physics include clicker questions, first popularized by Mazur [31] using the Peer Instruction method. These are conceptual multiple-choice questions presented to a class for students to answer anonymously, typically individually first and again after discussion with peers, and with immediate feedback to instructors using an electronic response system, generally referred to as clickers [31]. They have proven effective and are relatively easy to incorporate into a typical course, without the need to greatly restructure classroom activity or assignments [32].

While these multiple-choice questions can be successfully implemented in physics classrooms without additional technological tools, this research used clickers, which automatically tracked student responses in real time. When presented in sequences of validated questions on a particular topic, clicker questions can systematically help students with particular concepts that they may be struggling with. Previously, such multiple-choice question sequences, or Clicker Question Sequences (CQS), related to several key QM concepts have been developed, validated and implemented [33–36]. Here, we describe the development, validation, and implementation of a CQS intended to help students learn about outer products and change of basis in two-state quantum systems.

II. METHODOLOGY

The developed, validated and implemented CQS is intended for use in upper-level introductory QM courses. The data presented here are from administration in a mandatory first-semester junior-/senior-level QM course at a large research university in the United States. During the development and validation process, we took inspiration from a previously-validated QuILT and CQS on Dirac notation, since much research involving cognitive task analysis had been conducted from both student and expert perspectives [28]. We also drafted and iterated new questions for the basics of two-state systems, including outer products and change of basis. To ensure that the material could be completed in the allotted class time, while offering maximal value to students, we prioritized the coverage of conceptual knowledge, used common difficulties as a guide, and provided checkpoints that could stimulate useful class discussions. We iterated the questions many times amongst ourselves and with other faculty members to maximize clarity, consistency, and accuracy. We then conducted think-aloud interviews with five students in order to fine-tune the questions and ensure that they are at an appropriate level.

The learning objectives of the CQS are threefold. CQS 1.1-1.4 help students identify properties of two-state spin systems and spin-1/2 systems in particular. CQS 2.1-2.6 help students achieve fluency in translating between Dirac notation and matrix representation and calculating outer products. Finally, CQS 3.1-3.5 build on this prior knowledge to help students change the basis of a quantum state through several approaches. In particular, the three methods discussed to help students be able to change basis were direct substitution (e.g., using $|\pm x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle)$ to replace a state expressed in the x -basis using the standard notation); the viewing of inner products, e.g., $\langle \pm x | \chi \rangle$, as projections of the state $|\chi\rangle$ along the basis vectors in a particular basis (in this case, the x -basis); and the use of spectral decomposition of unity. Here, we report on the results of selected concepts, namely outer products and change of basis, for which students had their difficulties greatly reduced. The clicker questions relevant to these concepts are provided below.

CQS 2.2 Given that

$$\begin{aligned} | +x \rangle &= \frac{1}{\sqrt{2}} (|+z\rangle + |-z\rangle), \\ | -x \rangle &= \frac{1}{\sqrt{2}} (|+z\rangle - |-z\rangle), \\ | +y \rangle &= \frac{1}{\sqrt{2}} (|+z\rangle + i|-z\rangle), \\ \text{and } | -y \rangle &= \frac{1}{\sqrt{2}} (|+z\rangle - i|-z\rangle), \end{aligned}$$

choose all of the following statements that are true:

- I. $\langle +z | +z \rangle = 1$ and $\langle -x | +x \rangle = 0$
- II. $\langle +y | +y \rangle = i$ and $\langle +z | +y \rangle = \frac{1}{\sqrt{2}}$

III. $\langle +x|+x\rangle = 1$ and $\langle +x|-y\rangle = \frac{1}{2}(1 - i)$

- A. I only B. I and II only **C. I and III only**
 D. All of the above E. None of the above

CQS 2.5 Use the following matrix representations

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Choose all of the following statements that are true:

- I. $|+x\rangle\langle -x| = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$
 II. $|+x\rangle\langle -x| = \frac{1}{2}(1 - 1 + 1 - 1) = 0$
 III. $|+z\rangle\langle +z| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- A. I only** B. II only C. III only
 D. I and III only E. None of the above

CQS 2.6 Use the following matrix representations

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |-x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$|+y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad |-y\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Choose all of the following statements that are true:

- I. $\langle +x|+x\rangle = \frac{1}{\sqrt{2}} (1 \ 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$
 II. $\langle +x|+x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} (1 \ 1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
 III. $|+z\rangle\langle -x| = (1 \ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}$
 IV. $|+z\rangle\langle -x| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} (1 \ -1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

- A. I only B. I and III only **C. I and IV only**
 D. II and III only E. II and IV only

CQS 3.1 A generic state is written in the $\{|+z\rangle, |-z\rangle\}$ basis as $|\chi\rangle = a|+z\rangle + b|-z\rangle$, but it can be written in another basis. In the $\{|+y\rangle, |-y\rangle\}$ basis, $|\chi\rangle = a'|+y\rangle + b'|-y\rangle$. Choose all of the following statements that are true:

- I. $a = \langle +z|\chi\rangle$ and $b = \langle -z|\chi\rangle$
 II. a' and b' are the projections of $|\chi\rangle$ along $|+y\rangle$ and $|-y\rangle$, respectively.
 III. $a' = \langle +y|\chi\rangle$ and $b' = \langle -y|\chi\rangle$

- A. I only B. II only C. I and III only
 D. II and III only **E. All of the above**

CQS 3.2 A generic state is written in the $\{|+z\rangle, |-z\rangle\}$ basis as $|\chi\rangle = a|+z\rangle + b|-z\rangle$. Given that $|+y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle + i|-z\rangle)$ and $|-y\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - i|-z\rangle)$, choose all of the following statements that are true:

- I. $|+z\rangle = \frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle)$ and $|-z\rangle = -\frac{i}{\sqrt{2}}(|+y\rangle - |-y\rangle)$
 II. $|\chi\rangle = \frac{a}{\sqrt{2}}|+y\rangle + \frac{ib}{\sqrt{2}}|-y\rangle$

III. $|\chi\rangle = a \left[\frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle) \right] + b \left[-\frac{i}{\sqrt{2}}(|+y\rangle - |-y\rangle) \right]$

- A. I only B. II only C. III only
 D. I and II only **E. I and III only**

CQS 3.3 One way to accomplish the process of writing a state in a basis is by acting upon it with the operator \hat{I} written in that basis. Choose all of the following that are valid statements:

- I. $\hat{I} = |+x\rangle\langle +x|$
 II. $\hat{I} = |+x\rangle\langle +x| + |-x\rangle\langle -x|$
 III. $\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in any basis chosen
 IV. $\hat{I}|\chi\rangle = |\chi\rangle$

- A. I and III only B. III and IV only
 C. I, III, and IV only **D. II, III, and IV only**
 E. All of the above

CQS 3.4 Given that the identity operator \hat{I} multiplied by any state returns that state, choose all of the following that are an equivalent way of writing the state $|\chi\rangle$:

- I. $|\chi\rangle = \hat{I}|\chi\rangle = (|+x\rangle\langle +x| + |-x\rangle\langle -x|)|\chi\rangle$
 II. $|\chi\rangle = \hat{I}|\chi\rangle = (|+y\rangle\langle +y| + |-y\rangle\langle -y|)|\chi\rangle$
 III. $|\chi\rangle = \hat{I}|\chi\rangle = (|+x\rangle\langle +x| + |-y\rangle\langle -y|)|\chi\rangle$

- A. I only B. II only C. III only
D. I and II only E. All of the above

CQS 3.5 If we want to express our state in the x -basis, we can write $|\chi\rangle = \hat{I}|\chi\rangle = (|+x\rangle\langle +x| + |-x\rangle\langle -x|)|\chi\rangle$. Choose all of the following that correctly represent the state $|\chi\rangle = a|+z\rangle + b|-z\rangle$ in the $\{|+x\rangle, |-x\rangle\}$ basis:

- I. $|\chi\rangle = (|+x\rangle\langle +x| + |-x\rangle\langle -x|)(a|+z\rangle + b|-z\rangle)$
 II. $|\chi\rangle = |+x\rangle\langle +x|a|+z\rangle + |-x\rangle\langle -x|b|-z\rangle$
 III. $|\chi\rangle = a|+x\rangle + b|-x\rangle$

- A. I only** B. II only C. III only
 D. All of the above E. None of the above

The final version of the CQS was implemented once online and twice in person. The CQS during the online implementation was presented as a Zoom poll with questions displayed via the “Share Screen” function, and for the in-person implementation, the poll was replaced by a functionally similar classroom clicker system. For each question, after displaying the polling results, the instructor held a full class discussion of the possible options provided. The Peer Instruction component was present in the in-person administration, but not the online administration because of difficulties in fostering small-group student discussion in the online environment. We note also that the instructors were different for the online and in-person classes.

To determine the effectiveness of the CQS, we developed and validated a pre- and post-test containing questions on topics covered in the CQS. The post-test contained small

changes in numerical values and the basis in which some states were expressed, but was conceptually similar. In both online and in-person classes, students completed the pre-test immediately following traditional lecture-based instruction on the topic. After administration of the CQS, students completed the post-test. Two researchers graded the pre-test and post-test, and converged after discussion on a rubric with inter-rater reliability greater than 95%. Selected pre- and post-test questions are reproduced below; some other questions are not examined here due to limited space. Question Q1 asks students to calculate the outer product given two ket states, which requires finding the corresponding bra state for one of them. Questions Q2 and Q3 ask students to express a given state in a different basis.

Q1. Given that in the z -basis

$$|\chi\rangle = \frac{1}{\sqrt{34}} \begin{pmatrix} 3i \\ 5 \end{pmatrix} \text{ and } |\psi\rangle = \frac{1}{\sqrt{17}} \begin{pmatrix} 4 \\ -i \end{pmatrix}$$

Find $|\psi\rangle\langle\chi|$. Show or explain your work to get credit.

Q2. Write the state $|\chi\rangle = \frac{3}{\sqrt{10}}|+x\rangle + \frac{1}{\sqrt{10}}|-x\rangle$ in the form $|\chi\rangle = a'|+z\rangle + b'|-z\rangle$ (that is, find a' and b'). Show or explain your work.

Q3. Write the state $|\chi\rangle = \frac{3}{\sqrt{10}}|+z\rangle + \frac{1}{\sqrt{10}}|-z\rangle$ in the form $|\chi\rangle = a'|+x\rangle + b'|-x\rangle$ (that is, find a' and b'). Show or explain your work. Note that we start here in the z -basis and wish to convert to x -basis, the opposite of the preceding question.

III. IMPLEMENTATION RESULTS

Though this study is quasi-experimental in design [37], prior work has shown that, compared to control groups given traditional instruction and homework, students had significantly better post-test performance when engaging with research-based tools [29,38]. The pre-test scores, post-test scores and effect sizes [39] are listed in Tables I-III. Table I gives results for the online class ($N = 29$), and Tables II-III do so for in-person classes 1 ($N = 25$) and 2 ($N = 27$). Overall, the results are encouraging, and students performed well on the post-test, with reasonably impressive effect sizes. Below, we discuss some difficulties that were addressed after the administration of the CQS for all three years.

A. Outer product

Question Q1 asks students to calculate the outer product of two given states. Students' most common mistake was to provide a scalar rather than a matrix, for which zero credit was given. This was observed for many students' pre-test responses in all classes, and some students' post-test responses during the online class. Students in the in-person years performed better on the post-test, with some even writing their answers in Dirac notation, indicating that these students were comfortable translating between the Dirac

notation and matrix representation. The majority of students preferred using matrix representation, likely because the question had the given states in matrix representation and because it offers more compact notation. Some students neglected to take the complex conjugate when finding the corresponding bra state. Additionally, some students found the bra state corresponding to the ket state other than the one indicated, but they were given full credit if they otherwise performed the outer product correctly. Among incorrect responses, it was also common for students to provide the transpose of the correct matrix. On a related note, some provided the correct answer but showed for their work a nonstandard method for matrix multiplication, placing the row matrix to the left of the column matrix (which conventionally yields a scalar), indicating that they knew the result of such an operation but had not mastered the mechanics of matrix multiplication. It is possible that those students who unintentionally transposed the matrix also did not have a full understanding of the rules of matrix multiplication. All of these difficulties were much more prevalent on the pre-test than the post-test for all years.

B. Changing from x -basis to z -basis

Question Q2 provided a state in the x -basis and asked students to change to the z -basis. This question was deliberately left open-ended so that students could use the method with which they were most comfortable, as the CQS covered several distinct approaches to changing basis. Most students chose to directly substitute $|\pm x\rangle$ states with their expressions in the z -basis, $\frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle)$, which were given in the pre-test and post-test.

The most common mistake on the pre-test, other than a few students who left the question blank or made little progress, was to simply divide the expansion coefficients in the starting basis by $\sqrt{2}$. Students with this type of difficulty did not recognize that such a state is not normalized. While the details of how students arrived at this result varied between students, in many cases it may stem from discarding some of the inner products $\langle+x|+z\rangle$, $\langle+x|-z\rangle$, $\langle-x|+z\rangle$, and $\langle-x|-z\rangle$ in their attempts to obtain the final answer, resulting in an incomplete projection along the new basis. Other students arrived at the conclusion that the expansion coefficients do not change when transforming from one basis to the other (i.e., $|\chi\rangle = a|+x\rangle + b|-x\rangle = a|+z\rangle + b|-z\rangle$). These notions were largely corrected on the post-test.

Additionally, on the post-test, rather than finding $a|+x\rangle + b|-x\rangle$ in the z -basis as asked, one student was observed to instead find the state $b|+x\rangle - a|-x\rangle$, which is orthonormal to the given state. This was an interesting response, as it demonstrated an understanding of what makes an orthonormal basis, and possibly the idea that one can arbitrarily construct an infinite number of bases, though it had little to do with transforming from the given basis to the target basis.

TABLE I. Results of the online administration of the CQS via Zoom (online class). Comparison of pre-test scores, post-test scores and effect Cohen's d (calculated by $\frac{(\mu_{post}-\mu_{pre})}{\sqrt{\sigma_{pre}^2+\sigma_{post}^2}}$) [39], for students who engaged with the CQS ($N = 29$).

Q#	Pre-test	Post-test	d
Q1	47%	60%	0.31
Q2	69%	84%	0.37
Q3	50%	86%	0.86

TABLE II. Results of the first in-person administration of the CQS (in-person class 1). Comparison of pre- and post-test scores, along with effect size as measured by Cohen's d , for students who engaged with the CQS ($N = 25$).

Q#	Pre-test	Post-test	d
Q1	66%	94%	0.91
Q2	92%	98%	0.29
Q3	62%	98%	1.08

TABLE III. Results of the second in-person administration of the CQS (in-person class 2). Comparison of pre- and post-test scores, along with effect size as measured by Cohen's d , for students who engaged with the CQS ($N = 27$).

Q#	Pre-test	Post-test	d
Q1	59%	85%	0.69
Q2	61%	87%	0.62
Q3	48%	78%	0.65

C. Changing from z -basis to x -basis

Like question Q2, question Q3 presented a state for students to change to another basis, this time starting in the z -basis and going to the x -basis. However, the relationships $|\pm z\rangle = \frac{1}{\sqrt{2}}(|+x\rangle \pm |-x\rangle)$ were not explicitly provided. Since it is possible to add and subtract the $|\pm x\rangle = \frac{1}{\sqrt{2}}(|+z\rangle \pm |-z\rangle)$ equations to find the explicit relationships, direct substitution is still a viable method. Some students went through this algebra, while others correctly recognized that the relationship between the x -basis and z -basis is symmetrical, which was another valid justification. These extra algebraic steps, however, can be challenging for some students and cause cognitive overload as they try to process all of the information, so question Q3 was considered to be more difficult than question Q2. Illustrating this, on the pre-test, many students (more than for Q2) were unable to make much progress with this question. In a similar vein, several students were observed to state that $|\chi\rangle = a|+x\rangle + b|-x\rangle = a|+z\rangle + b|-z\rangle$, with the expansion coefficients being the same in both bases, despite successfully working out the basis change for Q2. In all classes, a great majority of these students arrived at a correct answer on the post-test. Overall, for both questions Q2 and Q3, substitution was the dominant method, and fewer students used the spectral decomposition of the identity operator or explicitly described their work in terms of taking projections along the new basis, which could

point to a lesser emphasis on these methods in class and on homework assignments.

IV. COMPARISONS BETWEEN ONLINE AND IN-PERSON IMPLEMENTATIONS

The pre-test scores were around 50-60% for all years, with some exceptions, most notably Q2 during in-person class 1. Similarly, with the exception of Q1 for the online class, the post-test scores were all quite high, at about 80-90%, with medium to large effect sizes for questions with greater improvements (see Tables I-III).

The biggest difference in this regard is observed in question Q1 during the online class, which is the only one that did not feature Peer Instruction. This question was on outer products, and Q1 scores for this class were lower than they were in the others. This could have been a concept that was more difficult for students to grasp, thanks to the difficulties posed by the online environment. Despite this, however, they went on to quite proficiently change the basis of two states in Q2 and Q3 after the CQS (see Table II). It is possible that the CQS was more effective at helping students learn change of basis than outer products in an online format, or that higher-quality in-person discussions helped students better understand outer products.

Online instruction results seem to be only slightly worse or no worse than in-person results. This is consistent with what we have seen in other studies [35,36]. One possible reason the online results without peer instruction are not significantly worse than the in-person results with peer instruction is that students had their cameras off; as such, some may have consulted resources even though they were not supposed to consult any [40].

V. SUMMARY

Validated clicker question sequences can be effective tools when implemented alongside classroom lectures. We developed, validated, and found encouraging results from implementation of a CQS on the topic of change of basis for two-state quantum systems. We find that students tend to struggle with outer products after traditional lecture-based instruction, typically trying to find a scalar or getting lost in non-standard matrix multiplication schemes. When changing basis, common mistakes included simply dividing the state by $\sqrt{2}$ or answering that the expansion coefficients are the same in both bases. These difficulties were much reduced on the post-test.

Following the administration of the CQS, post-test scores improved across both instructors and both modes of instruction, showing that the CQS can be helpful in a variety of situations. It is thus likely that the CQS can help reduce student difficulties on these concepts for any instructor who is interested in implementing it in their own courses.

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