Investigating Students' Understanding of Entanglement

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Entanglement is at the heart of emerging technologies such as quantum computing, quantum sensing, and quantum networking. These technologies are developed by interdisciplinary teams of engineers, physicists, software developers, and other scientists. As a result, quantum concepts, such as entanglement, are essential to introduce to a wide range of disciplines. In response, many universities have implemented quantum information science and engineering (QISE) courses for undergraduate students. These QISE courses present quantum concepts to students from diverse disciplines at an introductory level, whereas previously, they were reserved for advanced physics courses. We interviewed students who had recently completed an introductory QISE course to investigate their conceptual and mathematical understanding of entanglement. We found that entanglement was most strongly associated with the idea of correlation between measurements, while only some students productively used the mathematical criterion of whether or not a quantum state was factorable.

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I. INTRODUCTION

Quantum physics has its origins in the study of atomic spectra, molecular structure, and properties of materials. While originally a model for explaining how nature works, quantum mechanics has gradually become more significant for developing technologies. Quantum mechanics is essential to understanding the device physics and the behavior of diodes, transistors, lasers, and nuclear magnetic resonance. Specifically, the operation of these devices relies on controlling quantum states of ensembles of particles, not individual quantum states. A second quantum technology revolution is underway, where the goal is to develop systems of interacting and individually controllable quantum systems that demonstrate significant advantages over classical technologies. For instance, quantum computers consist of quantum bits (i.e., qubits) that can be controlled and can interact with other qubits to form quantum logic gates and execute algorithms. Other appplications include the development of quantum sensors and quantum networks. One key feature of many of these emerging quantum technologies is that they rely on the use of entangled quantum states as a resource to improve beyond classical performance.

With the advent of quantum technologies, entanglement has become a core concept within quantum information science and engineering (QISE) curricula. Because the development of quantum technologies requires diverse STEM disciplines [1, 2], engineers, computer scientists, chemists, and physicists may all benefit from deeper background knowledge in quantum concepts, such as superposition and entanglement. Expanding QISE education is a priority in both the United States [1] and Europe [2]. The concept of entanglement dates back to 1935 with a thought experiment known as the Einstein-Podolosky-Rosen (EPR) paradox [3, 4]. Decades later, the thought experiment was realized in the laboratory and has been recognized with the 2022 Nobel Prize in Physics for being a key milestone in the progress toward quantum technology [5]. Within undergraduate physics education, entanglement was more of a curiosity and, prior to quantum computing, was not seen as a central concept. In quantum textbooks predating 1990, the term entanglement was unlikely to appear in the index. If entanglement was mentioned, it was typically associated with the EPR paradox and a more philosophical discussion about locality, realism, hidden variables, and the nature of quantum mechanics. Within secondary education, the introduction of quantum concepts tends to follow a more historical progression, and entanglement is generally not included in educational standards [6]. However, recent frameworks for high school quantum information science education provide detailed learning objectives that include entanglement [7]. There is a gap between the older philosophical presentations of entanglement and modern technological applications, and there is a need to introduce the concept to an audience beyond physics majors. All of this motivates a need to better understand how students learn entanglement.

Within literature aimed at QISE, entanglement is commonly introduced alongside its significance for providing a quantum advantage to computational algorithms, teleportation, communication, and information retrieval. Furthermore, we found that upper-level textbooks in quantum computing and quantum physics often used a thought experiment (extending the EPR paradox) involving communicating and sending information between two people (e.g., Alice and Bob) to introduce entanglement of a two-particle system [8–10]. Introductory resources intended for an audience with little to no experience in quantum or physics used analogies relating classical systems to quantum entanglement: pairs of quantum coins [11] or pairs of quantum gloves [7].

While there is substantial prior research on students' understanding of quantum mechanics (see the review by Michelini and Stefanel [12]), there is not much work on entanglement specifically. One study, as part of the Quantum Mechanics Visualization Project, investigated students' understanding of entanglement after completing a computer simulation activity that explored measurement outcomes for two-particle spin states with varying input states [13]. The post-test examined students' ability to recognize entangled states and a conceptual question related to the simulation that explored the relationship between correlation in measurement outcomes and whether or not states were entangled. We wanted to advance the research on knowledge of students' ideas about entanglement by examining the concepts that students drew on to explain entanglement and how those ideas were deployed to answer a mathematical question. While we did not set out to identify a list of resources (e.g., Ref. [14]), our work is done within that framework [15]. We seek to answer two questions: (1) How do students describe entanglement and qualitatively differentiate entangled states from non-entangled states? (2) How do students recognize if a state is entangled or not based on its mathematical form?

II. METHODS

To investigate students' understanding of entanglement in a multidisciplinary setting, we needed a working definition of entanglement that was accurate and reflected definitions that might be given at a range of levels of complexity. We synthesized the discussions from 12 textbooks and online resources that spanned various difficulty levels, from beginner [7, 11, 16–20] to advanced [8–10, 21, 22], and that focused on general quantum mechanics [7–11, 16, 21] and quantum computing [7, 17–20, 22]. Our definition of entanglement is *an inseparable relationship between two or more quantum objects, in which the measurement of one will be correlated to the other, and which is created by an interaction between the systems*. This definition has three features we focused on later in the analysis: correlation, measurement, and an inseparable relationship between multiple quantum objects. Our analysis did not include the relationship between interaction and entanglement because that relates to how entanglement is created rather than distinguishing entangled from non-entangled states.

To understand students' ideas about entanglement, we conducted 7 interviews with undergraduate STEM majors during the summer after they had completed one or more undergraduate courses in QISE. An undergraduate researcher conducted the interviews to increase participants' comfort and develop the researcher's interviewing skills. The interview consisted of three sections and lasted between 20 and 38 minutes. The first section investigated the students' background, including their year of study, major, prior experience and interest in quantum mechanics, and general feedback on their QISE courses. The second section probed students' conceptual understanding of entanglement in multiple complementary ways by asking students to: describe entanglement in their own words, explain the difference between entangled and non-entangled states, explain the significance of entanglement, and provide analogies that came to mind when they thought of entanglement. The third section examined students' mathematical understanding of entanglement by presenting entangled and non-entangled states in Dirac notation, a standard notation used in quantum mechanics. Students were then asked to identify which states were entangled and explain their reasoning, which gave an opportunity to apply their ideas about correlation, inseparability, and measurement. Since Dirac notation differs between courses and textbooks, we presented students the most common options from the twelve sources, including two different notations for the state ($|\uparrow\rangle$, $|\downarrow\rangle$ or $|+\rangle$, $|-\rangle$) and three different notations for product states ($|\uparrow\rangle \otimes |\downarrow\rangle$, $|\uparrow\rangle|\downarrow\rangle$, or $|\uparrow\downarrow\rangle$). Students chose the notation they were most familiar with, and all subsequent math questions used their preferred notation. The seven students chose four different preferred notations, though most students were comfortable with any of the options. Students then answered four mathematical questions using pencil and paper as necessary and were prompted to explain their thought process and final answer verbally.

Since the interviews were conducted over the summer when classes were not in session, we could not administer the entanglement questions as homework or extra credit assignments. Instead, we opted for conducting interviews, which allowed students to articulate their reasoning in a 'think-aloud' format. This approach provided deeper insight into how students think about entanglement and the associated ideas of measurement, correlation, and inseparability. However, the interview method did limit the size of our data set. Additionally, we constructed our interview to slowly increase in difficulty, starting with basic descriptions and then moving to more complex mathematical questions.

Students were recruited from two introductory quantum information science courses (QIS) offered at Rochester Institute of Technology: Physics 251 Principles and Applications of Quantum Technology and Computer Engineering 257 Introduction to Quantum Computing and Information Science. Each introductory course had minimal prerequisites, leading to a diverse class population in terms of year and major. We sent recruitment emails to students who had taken either course in the past year. Seven participants were interviewed: four 5th-year students (in majors that required cooperative learning experiences), two 4th-year students, and one 2nd-year student. Regarding their majors, there were three physics, two computer engineering, one computer science, and one computing and information science. This group represents 26% of the total combined enrollment. It is important to note that several students were enrolled in both courses while others only took one.

The audio for each interview was recorded with Zoom, transcribed by an online automatic transcription service (Otter), and manually edited to fix errors. To analyze the transcribed data, we first segmented each interview by question and the corresponding discussion. For instance, in response to the question, "How would you define entanglement in your own words?" we created initial descriptive codes, which were brief phrases summarizing the concepts students used in their responses to the presented question. Following the initial coding, we conducted a comparative analysis of student responses to identify common themes and concepts. We then mapped these descriptive codes onto a predefined coding scheme based on our consensus textbook definition of entanglement, particularly the concepts of correlation, measurement, and the inseparable relationship between multiple quantum objects. A similar process was followed for the other questions (e.g., contrasting entangled and non-entangled stages, analogies, and quantum states in Dirac notation). Through this coding scheme, we could systematically identify productive ideas and sources of confusion related to students' ideas about entanglement. In the third section of the interview, which asked whether particular quantum states in Dirac notation were entangled or not, we characterized student responses based on their correctness and reasoning.

III. RESULTS

A. Student descriptions of entanglement

Our first research question explored how students defined entanglement. We compared their definitions to our consensus definition, which included three key features: correlation, measurement, and the inseparable relationship between multiple quantum objects. This comparison aimed to identify which concepts from the consensus definition the students drew on in their own definitions.

Student F, a physics major, defined entanglement as "when you have two quantum systems with measurement outcomes that are more strongly correlated than allowed classically." Student F demonstrates a strong definitional understanding of entanglement by correctly referencing correlation and measurement, addressing two of our consensus definition concepts. Furthermore, student F recognizes the necessity of taking multiple measurements to determine a correlation between parts of an entangled quantum system, going beyond our consensus definition. At this point in the interview, student F did not mention the inseparable relationship between multiple quantum objects.

Student A, a computer science major, defined entanglement as something “that is a little bit weird and also popularized in a weird way when two things are entangled, learning something about one means that you will for sure gain information about the other.” Student A referenced two concepts from our consensus definition: we interpret the phrase “learning something” as a reference to measurement, and correlation is implied by the phrase “gaining information about the other.” Interestingly, Student A, with a background tied more closely to computing and information science, chooses language linked to information rather than measurements and statistics. However, unlike the previous definition by Student F, Student A lacks clarity with specific quantum terminology and does not address how entanglement is different from classical correlations.

Students were then asked, “What distinguishes an entangled state from a non-entangled state?” This is where students began to reference the inseparable relationship between quantum objects as a component of their entanglement description. Four students (3 physics, 1 computer science) mentioned an inseparable relationship between two quantum systems. Two students (computer engineering and computer information technology) mentioned correlation. One student (computer engineering) stated they did not know. The physics majors had additional exposure to quantum-related topics through courses, such as quantum mechanics, which can be applied to understanding QISE concepts, such as entanglement.

Student A, a computer science major, referenced inseparability, stating, “We have an entangled state you can’t factor out into individual pieces because they’re related to each other. Whereas a non-entangled state you can factor each qubit in the system separately.” This response accurately addresses the inseparable relationship between quantum systems when they are entangled and invokes a mathematical example of whether or not you can factor a two-qubit state into the product of single-qubit states. Interestingly Student A’s initial definition of entanglement only mentioned correlation and measurement, however the follow up question of distinguishing entangled vs non-entangled states resulted in the additional criteria of inseparability. This example suggests that students may find the inseparability or factorability of quantum states as the aspect of the entanglement that connects most closely to the mathematical representation of quantum states.

Student C, a computer engineering major, did not invoke separability. They stated, “I remember we did a proof of it that I definitely can’t replicate right now. But more conceptually, just the fact that when one changes, the other will, even if nothing’s acting on it.” Student C doubts their ability to recreate a proof (perhaps meaning a mathematical criterion) that distinguishes entangled from non-entangled states and returns to the concept of correlation. Furthermore, their response uses vague language, referencing “one” and “other” rather than quantum states. Student C’s inability to reference inseparability indicates a partial conceptual understanding of entanglement.

Which of the following are entangled?

I. $|\Psi\rangle = |\uparrow\rangle \otimes |\uparrow\rangle$

II. $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$

III. $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle)$

IV. $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$

FIG. 1. Interview question used to address research question 2.

Our investigation into how students define entanglement revealed that they commonly drew on the correlation concept of entanglement. All participants identified entanglement as related to correlation, but only three mentioned measurement, and none referred to the inseparable relationship between multiple quantum objects. The concept of inseparability only emerged when students were asked to distinguish an entangled state from a non-entangled state.

B. Recognizing entangled states

We address our second research question related to students’ ability to recognize entangled states using their mathematical form by analyzing students’ answers and reasoning on the problem shown in Fig. 1. Students were presented with four different quantum states that involved two qubits and told to determine *which of the following are entangled*. This question aimed to see if students could recognize the inseparability of entangled states with factoring. States III and IV are entangled since they can not be factored. State I is not entangled because it is already written as a product state. State II is not entangled because the left qubit in both terms is in the spin-down state, which can be factored out from the right qubit, which is in a superposition state of spin-down and spin-up.

Two of the students were fully correct, accurately identifying both entangled states. Two other students were partially correct. Three students were fully incorrect or unable to answer the question confidently.

Student G, a computer science major, answered the question fully correct, identifying States III and IV as the entangled after spending time evaluating the problem on pen and paper. When Student G was asked why States III and IV were entangled, they said, “If you have either like two ones in the middle or on the top and bottom of a matrix representation of it. Then it cannot be formed via the tensor product of two normal vectors.” Student G could correctly apply the inseparability of quantum states to distinguish an entangled state from a non-entangled state. This is interesting since they did not mention the concept of inseparability in their prior conceptual responses describing entanglement. Other students, such as Student D, a physics major, when conceptually asked what distinguishes an entangled state, responded with, "it was really just that hey, not entangled state can be sort of broken up into, you know, separate tensor products. But an entangled state cannot." However, they failed to apply this reasoning to the mathematical problem, which was fully incorrect. Student D stated, "Again, I feel like I'm a little lost on the exact steps. But I probably say that State I is entangled. While the States II, III and IV are not." State II was a separable state and thus not entangled. Student D verbally described inseparably but was unable to apply it mathematically, which highlights another potential difficulty in understanding entanglement.

Student E, a physics major, was partially correct in their response, correctly labeling states II, III, and IV: "The second one is not entangled because it's separable. The third one is a Bell state, and the fourth one is as well. They're entangled." However, for State I, Student E explained, "[State I] is a little odd, right? Because you would measure particle one. And because it doesn't have any other components, you would know what particle two is. So I guess if I was relying on my deterministic definition, it would be entangled even though I don't like saying that." Student E correctly applied factorization to States II, III, and IV. However, Student E becomes confused when presented with State I. Productively, student E notices there will be a correlation between measurements of the two particles or qubits, but does not clarify that those correlations only exist for measurements made in the z-basis. If the measurements were made in the x- or y-basis, there would be no correlation. One source of confusion may be that State I looks similar to half of the entangled Bell State $1/\sqrt{2}(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle)$, which exhibits correlations between the two qubits when measured along any axis. Student E's response reveals the need for a nuanced understanding of when correlation implies entanglement and coordinating the concept of correlation with mathematical formalism and inseparability.

The results also showed that some students (G) could correctly apply the concept of inseparability to distinguish entangled states, but others (D) struggled to apply their verbal description of inseparability to mathematical representations of quantum states.

IV. DISCUSSION AND CONCLUSIONS

Our findings illustrate that the students in our study primarily defined entanglement in terms of correlation and rely on this concept when solving problems. We argue that students gravitate towards the correlation aspect of entanglement because most entry-level textbooks introduce the topic with a simple classically correlated system or communication before extending the discussion to quantum entanglement. Students' use of a description involving correlation is not incorrect, but should be supplemented by a nuanced description of which measurements and correlations can distinguish an entangled from a non-entangled state.

Additionally, in line with the resources framework, students' understanding of entanglement is context-dependent. Different questions evoked different aspects of entanglement, and coordinating all the ideas to form a coherent understanding of entanglement is tricky. Student F demonstrated a strong understanding of correlation and measurement but struggled to incorporate the concept of inseparability when defining entanglement. Students A and C faced the same difficulty. This suggests that students' early introduction to entanglement as correlation can be challenging to coordinate with the concept of inseparability.

Furthermore, when we presented students with a problem requiring the application of inseparability of quantum states, some students correctly applied idea of inseparability, while others struggled to coordinate the concepts of correlation and inseparability. Student G in Sec. III B exemplified the complexity of applying the correlation concept to mathematical representations of two-qubit states.

There is a complex network of concepts (e.g., two-qubit states, projective measurements in different bases, ensembles of repeated measurements, correlations, separability) that need to be integrated to make sense of entanglement, the EPR paradox, and Bell's inequalities. While some students showed proficiency in defining and recognizing entanglement, others misapplied ideas or were unsure. Even after a semester or more of an introductory quantum information science course, students may exhibit confusion about entanglement. Our results show similar difficulties to Kohnle and Deffebach [13], who found that students had difficulty recognizing entangled states based on separability. However, those students appeared to have a more robust set of ideas around measurement and correlations, possibly due to the online simulation tool that was used prior to the assessment.

Our results indicate a useful learning goal for entanglement would be the ability to evaluate two-qubit quantum states for entanglement using factorability and using measurements and correlation calculations. This would provide more nuanced understanding of the link between correlation and entanglement and help students coordinate the concepts of correlation and inseparability. Furthermore, students could be presented with states that are partially entangled, as Kohnle and Deffebach [13] suggest, to help students understand entanglement is not a binary (yes/no) feature of a quantum state, but there is a continuum of degrees of entanglement between fully entangled and fully separable.

Future work would benefit from using activities embedded within courses, which could examine students' ideas across a larger population. Future work could also explore students' views on how entanglement is produced between systems, including conceptual and mathematical models of physical processes that result in entanglement.

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