Student understanding of divergence and curl in upper-division electromagnetism

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The purpose of this paper is to understand and characterize student thinking about the physics concepts of divergence and curl in the context of upper-division electromagnetism for physics majors. We interviewed five students at a west coast university who had taken at least one semester of upper-division electricity and magnetism. The interview prompted students to describe divergence and curl in multiple representations (visually, mathematically, qualitatively). Our analysis identified that students can notice but struggle to resolve discrepancies between the mathematical calculations and the graphical representations of vector fields. We illustrate examples of when students appear to have a disconnect between these two representations and utilize ideas from related, but inappropriate representations. By providing examples of how students interpret divergence and curl both conceptually and mathematically, we hope to inform instructors of potential hurdles students may face when taking upper-division E&M.

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I. INTRODUCTION

Over the past decade, there has been a growing body of research on physics students enrolled in upper-division electricity and magnetism (E&M). This research has suggested that students in these courses can struggle to understand the conceptual meaning of various phenomena [1], demonstrate competency with Maxwell's equations [1], and relate mathematical tools from vector calculus to the physical applications [2, 3]. This prior work points to the importance of supporting students conceptually blending their knowledge of mathematics and physics [4, 5].

There have been several developments in this field in an attempt to resolve this issue, and this paper adds to this work by showing in-depth analysis of two student reasoning patterns. Our analysis of student understanding of specific topics in E&M is part of a broader project to build curricular materials to be utilized in the classroom.

Over the course of two semesters, we interviewed undergraduate and graduate students at a west coast university who had previously taken upper-division E&M but had not yet taken graduate-level E&M. Our goal was to determine how students relate mathematical concepts like divergence and curl to physical phenomena. We were specifically interested in whether students are able to understand the meaning of divergence and curl of vector fields without relying on the use of mathematical calculations.

In this paper, we will discuss two findings. We notice that some students are utilizing non-relevant topics to make sense of an explanation. More specifically, we see that students are pulling resources used in the integral form of Gauss's law when attempting to explain the differential form. We also notice that there appears to be some struggle for students to resolve inconsistencies between the visual representation and mathematical formulation of divergence and curl.

II. REVIEW OF LITERATURE

Prior research in upper-division E&M has discussed various difficulties students have with the material, as well as curricular innovations to support student learning. Interview-based studies have shown that students can struggle to understand concepts in the course, including knowing when to apply Gauss's law, distinguishing between electric flux and field, and noticing symmetry and superposition [1]. Additional research emphasizes how the use of vector calculus can be a challenge for students in the course, as competency requires blending one's understanding of vector calculus with physics concepts [3]. This prior work has led to the development of diagnostic tools that measure students' conceptual understanding and problem solving approaches [6, 7] as well as curricular innovations [4, 8].

We expand upon these prior investigations in how students think about upper-division E&M in a new institutional context. Our work is informed by a resources perspective on student thinking and learning [9, 10]. As described by Robertson et al. [10] within a resources perspective, "students' existing ideas are framed as continuous with (and as the basis for) more sophisticated or canonical understandings." Thus, the goal of our research is not to identify mistakes in a student's reasoning for the purposes of correcting it. Rather, we seek to characterize how students are making sense of the material and identify productive building blocks in students' thinking that can be generative for their future learning.

III. METHODS OF RESEARCH

The research conducted in this paper is a part of a larger study focused on development of curricular materials. Due to length restrictions, we will only discuss findings from participant interviews. Under the supervision of our institution's IRB, we recruited both undergraduate and graduate students to be participants for our research. The total number of participants in our research was five: four graduate and one undergraduate. Our requirement was that participants had to have completed a course in upper-division E&M and had not taken the graduate equivalent. After students had agreed to participate in this research, we conducted interviews either in-person or virtually. These interviews were intended to be a means of gauging student knowledge and retention of E&M after taking a course.

A. Interview protocol

The introduction of the interviews is meant to provide the interviewer some background knowledge on the participants' experience in their upper-division E&M course (e.g., "When did you take E&M?", "What was your course experience like?"). Participants were also asked questions regarding what they found difficult in the course, and how much vector calculus knowledge they are able to recall.

After establishing a background on the participants' experiences in E&M, the interview protocol asks about divergence in the context of E&M. Participants are first asked questions on "meaning of divergence" and to describe the meaning of Gauss's law in differential form. This provides insight on the initial ideas students bring to thinking about divergence, including their qualitative understanding. Once these questions have been completed, the participant is then asked to apply these concepts to electromagnetism.

Next, participants are asked to sketch and write down the formula for the electric field of a single point charge. The idea is to have participants display information in two different representations, and then have them analyze properties of each. Once students sketch the field of the point charge, they are asked about the divergence of said field. Afterwards, participants then calculate the divergence mathematically from the electric field that they wrote down. Once participants have obtained an answer mathematically, they are then asked to compare their answers about the divergence from their sketch and their equation.

Similar questions are asked at the beginning of the curl section, such as asking what the "meaning of curl" is, still allowing both qualitative and quantitative answers. Participants are then asked when the curl of an electric field is zero.

Next, the interview protocol asks participants to refer to their original sketch of the field lines of a point charge. They then are asked to add a charge equal and opposite to the initial charge (creating an electric dipole). Once a sketch of the field lines is created, the interviewer asks them about the properties of the curl in this configuration. Once participants have discussed the curl from their sketch with the interviewer, they then mathematically calculate the curl of this configuration. After obtaining an answer, participants were then asked to compare their result with their interpretation of the vector field.

B. Analytical approach

After collecting interviews, the first author watched and content logged every interview to briefly summarize the topics and discussion. The first author then developed a set of analytic memos analyzing common reasoning patterns and comparing across participants. These analytic memos were iteratively revised with feedback from the second author. For example, one analytic memo was dedicated to student interpretation of divergence, with relevant data being included and analyzed in further detail. In this paper, we show the two reasoning patterns that occurred in more than one interview.

IV. FINDINGS

A. Uses of resources of different forms of Gauss's law

In this section, we discuss one form of logic that students used to understand Gauss's law in differential form. We found that out of five participants interviewed, two of them used resources from Gauss's law in integral form to make sense of the differential form. In the interview, participants are first asked to describe divergence qualitatively. Next, they are asked to describe Gauss's law in differential form. When one interview participant (we call Student A) was asked to describe Gauss's law in differential form, we saw evidence that they were using ideas from the integral form to make sense of the differential form of the equation. After the student and interviewer have written down Gauss's law in differential form on a shared whiteboard, the following exchange occurs:

Interviewer: Can you describe to me Gauss's law in differential form?

Student A: If you have a charge [student sketches the thick dot in the lower right] and you're looking into this [student sketches a dotted line circle enclosing the thick dot] area, this ρ would be the charge enclosed in the area you're looking at.

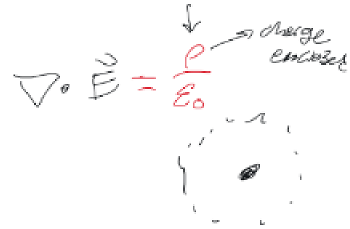

FIG. 1. Student work associated with conversation. Student writing is in black while the interviewer writing is in red.

We interpret Student A's explanation to reflect the use of ideas from the integral form of Gauss's law with the differential form of Gauss's law. As we interpret Figure 1, the dark circle represents a point charge, and the student's dotted circle around the charge encloses the point charge. However, the idea of an enclosed charge (a charge enclosed within a surface) is used in the integral form of Gauss's law. The ρ in the differential form of Gauss's law does not refer to an enclosed charge, but rather a volume charge density (charge per volume). Though Student A does not explicitly say the phrase "Gaussian surface," we interpret the dotted circle to be a Gaussian surface, which is used in the integral form of Gauss's law. The student may have been drawing on these resources based on previous uses of the integral form of Gauss's law to calculate the electric field. However, only in the integral form of Gauss's law (not the differential form) terms such as "enclosed charge" and the process of drawing a Gaussian surface are utilized.

The use of a Gaussian surface in the differential form of Gauss's law also appears in a separate interview with a different participant (Student B) describing the divergence of a vector field. When asked to describe the differential form of Gauss's law, Student B states:

Student B: Here ρ is representing a charge density and this ϵ_0 is a constant. And then the divergence, I think it means, when we are given our electric field [Student B draws electric field of a positive point charge] and then, well when we draw our Gaussian shape [Student draws a circle around the field of a point charge], the faucet would be our charge density generating the electric field and the sink would be our Gaussian shape. I think that's what this is saying: taking the divergence of a field that we can measure and we can figure out how much charge is associated with that.

In this quotation, Student B utilizes reasoning from both the differential and integral form of Gauss's law to describe the divergence of an electric field. They seem to blend in both of these resources, stating that the Gaussian surface is like a "sink." We interpret their use of the terms "faucets" and "sinks," to refer to points of positive and negative divergence using the "source/sink" metaphor. As they describe, areas with positive divergence look like a "source," with field lines emerging from the point, while areas with negative divergence tend to look like sinks. They indicate that the point charge is a source of divergence, given that the point charge is the source of the electric field. We also interpret Student B's mention of a "Gaussian shape" to be pulled from the integral form of Gauss's law, as the typical process to calculate the electric field of a charge configuration is to utilize a symmetric Gaussian surface to enclose charges.

While resources from the integral form of Gauss's law are not directly utilized to describe the differential form, we believe that they are still productive tools for students. Having resources from both forms potentially gives students an opportunity to connect the two interpretations and thus improve their understanding.

B. Difficulty resolving in visual and mathematical representations

In addition to students blending the use of resources from integral and differential forms of Gauss's law, we also found that students struggled to resolve inconsistencies across multiple representations. After Student A was asked to describe the differential form of Gauss's law, the interviewer asked them to mathematically calculate the divergence of this point charge, with the point charge located at the origin. The student first recalls the formula for divergence in Cartesian coordinates. They then realize on their own that the calculation should be made in spherical coordinates because "the electric field changes with radius." After some discussion, the interviewer writes the formula for divergence in spherical coordinates. Once the student sees this formula they proceed with the calculation.

While calculating divergence the student cancels out r^2 in a term with r^2/r^2 (which is inside a derivative), and ultimately arrives at an answer of zero. Concluding the divergence of a point charge is zero would be correct except where $r = 0$; for $r = 0$, r^2/r^2 is undefined, corresponding to the location of the point charge. The student does not appear to notice this discrepancy when they cancel the r terms, so they do not notice that there is a divergence at the point charge.

The interview then asks Student A to compare their graphical answer with their mathematical calculation.

Interviewer: How do your answers compare?

Student A: So I guess that it doesn't really line up with the drawings, there's no charge enclosed. So yeah, I'm not sure if there was a math error on my part.

This discussion illustrates that the student productively noticed a disconnect between their mathematical calculation and conceptual reasoning. When asked to describe the meaning of the divergence, Student A gave their answer in the form of a mathematical formula. Later, when asked to describe the divergence of the sketch, they were unsure with how to proceed.

Student A demonstrated productive mathematical resources for solving for divergence. They mentioned that an electric field of a point charge has no angular dependence, thus only depends on the radial coordinate. Once they deduced this, they were able to calculate the divergence with little issue. The student did make an error when canceling the r terms; they did not consider the case where $r = 0$. When Student A was asked to compare their answers, they noticed the inconsistency between their calculation and their conceptual understanding, which was productive. Though they did not resolve the inconsistency during the interview, they could productively build from their prior thinking by seeking to resolve the inconsistency.

A similar event occurred in an interview about the curl of a vector field with another student (Student C). In this segment, the interviewer asks Student C about the formulation of the curl of a vector field:

Interviewer: Do you recall when the curl of an electric field is zero?

Student C: The curl of an electric field is zero when... I think it's when you don't have a changing magnetic field.

Interviewer: When would you have a non-zero curl of an electric field?

Student C: I guess when you have a magnetic field that changes over time otherwise you have pure divergence.

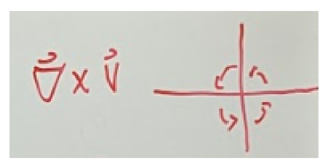

FIG. 2. Student C's description of Curl

In this interview, the student states that the curl of an electric field is zero when there is no changing magnetic field, accurately recalling an idea from electrostatics. The interviewer then asks the student to draw the electric field lines of an electric dipole. The student starts with a positive point charge and its field lines and is then asked to add a negative point charge and draw the new field lines. Student C draws a Cartesian coordinate plane and puts a positive point charge on the origin with field lines pointing outward in four diagonal directions. They then draw a negative charge on the positive vertical axis (Figure 3).

Interviewer: So if we add a negative charge like that what happens to the field lines?

Student C: Uh so all these field lines should start wrapping. [Student draws curved field lines starting at the positive charge and ending at the negative charge. The field looks like that of an electric dipole.] And these are gonna start wrapping around like that, which essentially adds, huh, that adds curl to it. I guess you don't need a changing magnetic field.

From drawing a diagram of an electric field of a dipole, the student interprets that there is a curl in this configuration. At the beginning of the curl section of this interview, Student C describes curl with a sketch of lines that curve around a fixed point (Figure 2) and they make circular, rotating gestures with their hands when they cannot find words to describe their explanation.

While sketching the field of a dipole, Student C sketches field lines that have curves. These curves of the field lines seem to be satisfactory for the student to conclude at that moment that there is curl in the electric field. The student then backtracks on their initial statement that there must be a changing magnetic field to have an electric field with curl. Within the case of the electric dipole, we see the student trying to resolve two ideas about curl—that curl is indicated by field lines with curvature or rotation, and that an electric field has curl when there is a changing magnetic field. As we see, the student chooses to agree with the visual representation.

Student C then proceeds to mathematically calculate the curl.

Interviewer: How would you calculate the electric field of this dipole?

Student C: Well, electric fields obey the law of superposition, so you can calculate the electric field of one of them and then just add it to the electric field of the other.

Student C draws on a productive resource to calculate the curl—the principle of superposition, meaning that the net electric field is the sum of the electric fields of the individual point charges. They then notice that each charge must be at a different location, hence them choosing distances r and $r + d$. After some dialogue with the interviewer about the formula for curl, Student C finishes their calculation. Student C notices that every term in their formula is zero, and they express their uncertainty why that is:

Student C: Why does every term go to zero?

Interviewer: How do these answers compare then?

Student C: I feel like I don't believe [student points to their equation for the electric field of a dipole] this. I know that electric fields obey the law of superposition. That's something I remember learning explicitly. I can calculate the electric field of one thing, calculate the electric field of another thing and just put them together. I'm pretty sure that's true, which leads me to believe that (the formula) is true. But also I look at this [student points to formula] and I look at that [student points to their drawing of the field lines of a dipole] and it doesn't feel like (the formula) reflects the picture I'm seeing there.

From their calculation, Student C (correctly) finds that there is zero curl for the representation. The student notices the inconsistency with their visual understanding of curl, an observation which would be a productive resource for problem solving, that could be further built upon.

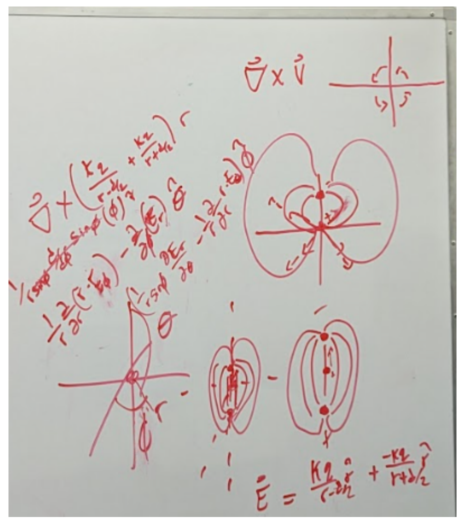

FIG. 3. Student C's sketch of field lines and calculations

V. CONCLUSION

This project used interviews to study how students in upper-division E&M think about divergence and curl of vector fields. We have seen that students do hold a variety of resources for making sense of these topics. We highlight two themes: (1) students can use ideas from the integral form of Gauss's law to understand the differential form and (2) students can notice, but struggle to bridge inconsistencies between conceptual ideas and mathematical formulation to solve problems.

We infer that the students' use of the integral form of Gauss's law stems from their exposure to it in both lower and upper-division E&M. If students utilize the integral form of Gauss's law more often than the differential form, they may be inclined to utilize the method with which they are more familiar. Furthermore, we see that interviewed students appear to notice inconsistencies between conceptual ideas and mathematical formulation for divergence and curl. We believe that it may be useful for instructors to develop curricular materials that provide students with an opportunity to reconcile different representations of divergence and curl.

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