Representation Issues: Using Mathematics in Upper-Division Physics

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Abstract. Upper-division students must learn to apply sophisticated mathematics from algebra, limits, calculus, multi-variable and vector calculus, linear algebra, complex variables, and ordinary and partial differential equations. The presenters in this session will discuss how the representations that we choose may affect whether students are able to use this mathematics spontaneously and correctly, whether they can move smoothly between representations, and the extent to which their understanding of the mathematics enhances their understanding of the physics. The discussant will incorporate the perspective of research in undergraduate mathematics education as it applies to the representations that have been presented.

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INTRODUCTION

Mathematics is ubiquitous in physics. It is a primary representation that can be used to articulate and demonstrate relationships among physical quantities. Mathematical facility allows a fuller understanding of empirical results, while more robust mathematical ability allows for the extension of physical concepts beyond a basic qualitative comprehension. Physics uses multiple mathematical representations within a given topical area, and specific mathematical concepts – and representations – are necessary for a complete understanding of the physics.

Beyond the introductory course sequence, students are expected to understand and apply increasingly sophisticated mathematics, including algebra, single- and multi-variable differential and integral calculus, vector calculus, linear algebra, differential equations, and complex variables. Perceptive instructors and curricular materials developers have noticed that it is not trivial for students to connect the mathematics they have been taught in pre- or co-requisite mathematics courses to the physics that they are learning [1,2].

Similarly, researchers in physics education at the upper division, in the course of their work, have gravitated towards investigations involving student understanding and application of the relevant mathematics [3]. In some cases they find that student difficulties with physics concepts may be related to issues with the mathematics, either an understanding of the math concepts, the application of the concepts to the physics, or in understanding some of the math representations used in physics.

In this session, we brought together colleagues who conduct research and develop curricula and curricular materials that focus on the role of representations in the use of mathematics in upper-division physics. The four presentations (please see the invited papers, also in this proceedings, as indicated) were Wittmann and Black [4] discussing integrals and limits in the context of classical mechanics; Price *et al.* [5] discussing vectors and vector fields in the context of electrostatics and magnetostatics; Thompson *et al.* [6] discussing partial derivatives in the context of thermodynamics; and Manogue *et al.* [7] discussing linear algebra in the context of quantum mechanics. After presentations and discussions of individual topics, we had a response from a member of the research in undergraduate mathematics education (RUME) community (JFW), to present his perspective on the research findings as well as his ideas about the relationship between these studies and what insight the RUME community could provide, and the extent to which these findings are of interest to the RUME community.

OBSERVATIONS ABOUT THE POSTERS IN THIS SESSION

Researchers in mathematics education have long recognized the role and significance of the use of multiple representations in revealing and supporting students' reasoning in mathematics, particularly as technology has dramatically increased the variety of representational forms available to support mathematical reasoning [8]. Indeed, the ability of representational forms to limit or distort student reasoning has also been well observed [9].

Representations Reveal Student Thinking

The papers in the present session all fit well within the continuing tradition of research on representations in mathematics learning, particularly as different representational forms reveal different aspects of students' thinking. Wittmann and Black found evidence of "more mathematical" and "more physical" thinking as two students handled even the simplest integral quite differently. Price, Gire, and Manogue showed how particular representational forms (including kinesthetic representations) could demonstrate limitations and flaws in student reasoning not readily accessible through standard textbook representations. Thompson *et al.* observed that processes of connecting mathematical and physical reasoning are not necessarily reversible. And Manogue *et al.* argued that introducing new representational forms to students can support student learning. All of these studies represent questions of interest within current mathematics education research.

Why Is This So Hard For Students?

Although few studies say so explicitly, a great deal of current work in mathematics education research explores the reasons behind students' difficulties with particular mathematical concepts. Between knowledge that "students find X difficult" and the temptation to create "instructional interventions to teach X," is a need to investigate the many good (but not at all obvious) reasons why some ideas so obvious to experts can be very challenging for students. Consider, for example, several questions related to Price, Gire, and Manogue's consideration of students' reasoning about the gradient [5,10]:

- Imagine an ant is crawling on the surface of a frying pan, and think of the gradient of the function that gives the temperature at any point on the frying pan. Show me a vector at the ant's location pointing in the direction of the gradient.
- Imagine the room is empty except for a burning candle placed at its center. Suppose your right arm is a vector pointing in the direction of the gradient of the function that describes the temperature at any point in the room. Which way does it point?
- Imagine an elliptical hill in the center of the room. Suppose your right shoulder is a point on the hill and your right arm is a vector pointing in the direction of the gradient of the function that describes the height of the hill. Which way does it point?

During the session at which these papers were first presented, several people in attendance opined that all three questions were essentially "the same." Further reflection reveals rather complex differences across them, however. In the first two questions, for example, the proposed location of the gradient vector (the surface of the pan and a point in space, respectively) correspond to the domain of the function generating the gradient. In the third question, however, the placement of the shoulder on the slope of the hill distracts from the fact that the gradient actually "lives" in the floor. Additional complexity arises from the limitation of the third question to entirely spatial dimensions. It can be quite natural to "point toward the direction in which temperature [a non-spatial variable] is increasing," but not at all obvious what it means to "point in the direction in which height is increasing."

Price, Gire, and Manogue also observed that "students often struggle to untangle the coordinates from the components [of vectors in vector fields]." Curiously, however, it seems that ambiguous representational practices within mathematics itself may encourage this struggle. A widely used calculus text (Stewart, 2012), for example, introduces a vector like this: "The term **vector** is used by scientists to indicate a quantity that has both magnitude and direction" [11] (*NB:* Location of the vector is not determined.) A few paragraphs later, it is observed that when the vector $\mathbf{a}$ is placed at the origin, "the terminal point of $\mathbf{a}$ has coordinates of the form (a_1, a_2) …These coordinates are called the **components** of $\mathbf{a}$ [12]." In the sections that follow, students learn that $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$ represents infinitely many vectors whose tails are all located at the origin of the Cartesian plane, while $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ represents infinitely many vectors, whose tails are located at all points in the Cartesian plane. Educators in mathematics and physics cannot be too surprised that students grow confused in their attempts to distinguish between vector coordinates and components.

At times, representational conventions within the physics community can place teachers at cross-purposes with each other. Thompson et al. observed that "some students treated differential expressions algebraically" rather than interpret them as total differentials. For example, when asked to find an expression for *B* given the total differential $dR = B\,dC + E\,dF$, students might begin by inappropriately treating dC as a variable and dividing:

$$\frac{dR}{dC} = B + E\left(\frac{dF}{dC}\right)$$

Curiously, during the same session, Wittmann and Black demonstrated a "correct" solution to a differential equation by casually treating a differential in precisely the same casual way:

$$m\frac{dv}{dt} = -bv^2$$
$$-\frac{m}{b}\frac{dv}{v^2} = dt$$

The use (and abuse) of representational forms throughout the physics and mathematics communities may appear to cynical students to be fiendishly designed to mislead them! At the very least, teachers and researchers might observe that practices such as these are often taken for granted but rarely discussed explicitly with students in a manner that addresses not simply notational convention but their underlying mathematical and physical implications and principles.

COMPARISON OF THE CURRENT STATE OF MATHEMATICS EDUCATION RESEARCH TO THE IDEAS OF THIS SESSION

As noted above, many of the research interests expressed by the participants in this session speak well to contemporary concerns of mathematics education research. From the perspective of a mathematics education researcher immersed in a conference of physics educators, however, a few points of comparison stood out as well.

Transfer Across Mathematics And Physics

Research on learning transfer has recently re-emerged as a fertile area of research in mathematics education. Surprisingly, however, very little research has been done concerning the transfer of mathematical ideas to other specifically academic disciplines. Although considerable work has been done with regard to the use of mathematics in everyday activities or within certain professions, most transfer research in mathematics education has focused on matters of transfer within mathematics itself. Wagner's [13] perspective on transfer would suggest that reasoning about the same mathematical ideas across multiple representations is a matter of transfer. In this light, studies suggesting that, under certain conditions, students may perform *better* on experientially grounded, real-world "story problems" than on corresponding algebra representations of the same problems [14] may be of particular interest to physics educators.

The work being done by physics education researchers on the use of mathematics in learning physics does not appear to be well known in the mathematics education community. The respondent at the current session is of the opinion that cross-disciplinary research on mathematics transfer deserves increasing attention in the mathematics education community, especially when one considers just how much mathematics is learned *while* students are learning physics and other mathematically rich academic disciplines.

Theoretical Commitments

One dimension of educational research that may be getting a bit more attention in the mathematics education research community is the contribution of discipline-specific educational research to learning theory. The theoretical commitments of researchers directly influence the questions they pose, the ways their data are gathered, and the interpretations that they permit of their findings. Consider several questions raised in this session:

- When a student sees $m\,dv/dt = -bv^2$, what does the student "see"?
- "Making connections between physics and mathematics is apparently not a reversible process?" Does this surprise us or not?
- After the students engage in the kinesthetic activities with vectors and they adjust their answers, what do we believe has happened to them?

Different epistemological commitments permit different answers to these questions. The reader's awareness (or not) of the researchers' epistemological commitments permits different interpretations of their conclusions. It is increasingly expected within mathematics education research that researchers and authors highlight their own theoretical commitments in their writing and extend their research, as possible, to highlight its implications for broader theory development.

The community of mathematics education researchers is far too broad to capture in just a few bullet points, but *among* the major areas of investigation in mathematics education these days are these:

- What can we reveal about student thinking that helps us understand why students struggle so much with particular issues?
- What do answers to that question suggest about potential instructional interventions? In other words, how best do we ground our intervention strategies in sound research?
- What do our empirical data tell us about current/competing theories of cognition, learning, and instruction?
- What is the interplay between individual cognitive aspects of learning and social/participatory aspects of learning?

The extent to which questions such as these are significant to the physics education community, as well, may be one measure of the degree to which increasingly collaborative research might prove beneficial to both disciplines.

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