

Scaffolding Students' Application of the 'Area Under a Curve' Concept in Physics Problems

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Abstract. We carried out several experiments in which we used sequences of physics problems to investigate students' ability to apply calculus concepts in physics problems. In this paper, we discuss an experiment which focused specifically on the concept of "area under a curve." We organized group problem solving sessions to teach students the concept of area under a curve using our problem sequences. We combined both a paper-based test and a computer-based test with online hints to assess students' ability to transfer their learning to solve new physics problems. We found that students' strategies for solving physics problems using this concept largely depend on problem type and scenario. Students' prior knowledge of area under a curve from calculus could interfere with their ability to learn a coherent model of using this concept in physics contexts.

Keywords: problem solving, area under a curve, transfer

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INTRODUCTION

Many physics problems present the graph of a function instead of its algebraic form, thus requiring students to solve the question graphically using the concept of area under a curve. Artigue [1] found that most students could perform routine procedures for finding the area under a curve, but few were able to explain these procedures. According to Thompson and Silverman [2], for students to perceive the area under a curve as representing a quantity other than area (e.g. displacement, work), it was important that they considered the quantity being accumulated as a sum of infinitesimal elements, each of which were formed multiplicatively. Recent research [3] has shown that when solving physics problems, students did not understand what physical quantity the area represented. Even though students could state that the integral equaled the area under a curve, they were unable to choose the correct graph to use.

In order to understand the area under a curve concept, it is important for students to relate the area to the structure of Riemann sums [4]. Consequently, understanding the concept of area under a curve requires students to recognize that area is accumulated from little bits of area, which represent a physical quantity derived from the multiplication of two other quantities.

Based on the work of Thompson and Silverman [2], we propose a three-step model for understanding and using area under the curve in physics contexts. The first step is to relate the area of a narrow rectangle under a curve to a simple physics equation (e.g. $\Delta x = v\Delta t$). The second step is to interpret the

accumulating process as adding up little bits of a physical quantity (e.g. displacement). The third step is to understand the approximation process, that is, breaking into more small rectangles would give a better approximation.

A tutorial sequence was designed based on this three-step model to help facilitate students' conceptual understanding of area under a curve in physical situations. We aim to address the following research questions:

1. Can students recognize the use of area under a curve in a novel physical context after training?
2. To what extent can students build the three-step model for understanding and using area under a curve?

METHODOLOGY

Experimental Design

A pretest-posttest comparison vs. treatment group, quasi experimental design was used. In the pretest students were presented a table with data about accelerations and velocities of a car at specific times and asked to calculate the total distance traveled. This problem could be solved by generating a velocity vs. time graph based on the given data followed by determining the distance travelled by estimating the area under the curve. The tutorial session was conducted several weeks after students completed exams on kinematics so they were familiar with the problem scenario. The purpose of the pretest was to

determine whether students could use area under a curve in a kinematics problem prior to our intervention.

During the second stage of the session, the treatment group was provided with research-based tutorial materials while a comparison group was provided homework problems covering the same topic - work done by a force. Students were encouraged to discuss their solution strategies with their partners while solving the problems. In order to better control the experiment, we did not give any verbal hints to either group. Instead, they were given printed solutions to each problem before proceeding to the next.

Nine problems were included in the tutorial material. The three-step model was implicitly embedded in the tutorial problems. The purpose of problems 1 through 3 was to remind students about the work done by a constant force, F acting over distance Δx in the direction of the force: $W = F \Delta x$ and to help them learn how to use unit multiplication. These three problems formed the first step of the three-step model. Problems 4 through 6 were designed to help them learn that the area of each small rectangle under the force vs. distance curve is a small amount of work, and small amounts of work can be added up. Problems 4-6 formed the second step of the model. Problem 7, which formed the third step, was designed to help students learn how to approximate the work when a continuous graph of force vs. distance is provided. Problems 8 and 9 were applications of the three-step model in the context of problems concerning energy and power to help students generalize the problem solving strategy across contexts [5]. Each problem was followed by three reflection questions [6]. The purpose of the reflection questions was to help students understand the learning goal of each problem and to strengthen the connection between the three steps. Students were not told the "three-step model" explicitly. Instead, they need to come up with the concepts on their own by solving tutorial problems.

The posttest problem, which was identical to the pretest problem except for a change in surface features was administered. First, students tried the problem by themselves and then used stepwise hints administered through the Mastering Physics online system. We designed hints to guide students to solve the problem by using the area under a curve concept.

Data Collection

Fifty-two students in a first semester calculus-based physics course participated in our tutorial session as an extra credit assignment. None had previously solved any physics homework or exam problems involving use of the area under the curve concept. However, the area under a curve concept was

covered in the prerequisite Calculus I course. The participants were randomly assigned to a treatment or comparison group for a tutorial session lasting 90 minutes. The pretest problem was attempted in the first 10 minutes. For the next 50 minutes, they worked on tutorial materials or selected homework problems in groups of three. Finally, they were asked to solve a posttest problem in ~25 minutes after which they were asked to explain the connection between the training problems and posttest. Their responses were coded to identify their reasoning of the area under a curve concept.

Data Analysis

Pretest, posttest, and tutorial worksheets were collected and the strategies employed by the students to solve the pre and posttests were coded. One common strategy employed was to apply a kinematics equation relating the different physical quantities. The other strategy was to sketch a graph and attempt to answer this question using the area under a curve concept. Therefore, common strategies were categorized as an equation or graph. If students did not provide enough information for us to identify the strategy they used, they were coded as "others".

For the equation strategy, we considered the correct solution was to apply a kinematics equation ($x = vt$ or $x = x_0 + v_0t + \frac{1}{2}at^2$) in each time interval in a correct manner and add the displacements. For the graph strategy, we considered the correct solution was to draw a velocity vs. time graph and then approximate the displacement as the area under the curve.

We also analyzed students' responses to the online hints. Students were required to write their thoughts and reasoning process about each hint. We also categorized the common justifications students provided to argue why displacement is the area under the velocity vs. time curve.

RESULTS

We compared strategies used by the two groups on the pretest and posttest as well as how these strategies changed after the tutorial session but before the students received the online hints. The results are presented in Table 1. One student in the treatment group used both strategies in the posttest, so in Table 1 the total number in one column does not equal the actual total number of students, N in treatment group.

In the pretest, more than 80% of students in both groups used the equation strategy and only a small portion of students recognized the use of the graph. We suspect that students were not familiar with using graphs in this course.

TABLE 1. Strategies used by students to solve the problems on the pretest and posttest.

Strategy	Pretest		Posttest	
	# of Students from Treatment Group (N = 31)	# of Students from Comparison Group (N = 21)	# of Students from Treatment Group (N = 31)	# of Students from Comparison Group (N = 21)
Equation	25	18	20	15
Graph	5	3	12	4
Others	1	0	0	2

For the posttest, seven more students in the treatment group used the graph strategy. In the comparison group, only one more student used this strategy. In general, more students in the treatment group successfully applied the graph method after they completed the tutorial material. However, about 60% of students in the treatment group still did not recognize the use of the graph after tutorial session.

The reasoning processes used by the students working through the online hints were coded in the following categories:

“Equation-Area relation” (EA) means students recognized that summing up the rectangle area under the curve was equivalent to summing up each $v \Delta t$. This type of reasoning indicates that students connected the “area under a curve” concept with a physical situation. It also corresponded to the first two steps of three-step model.

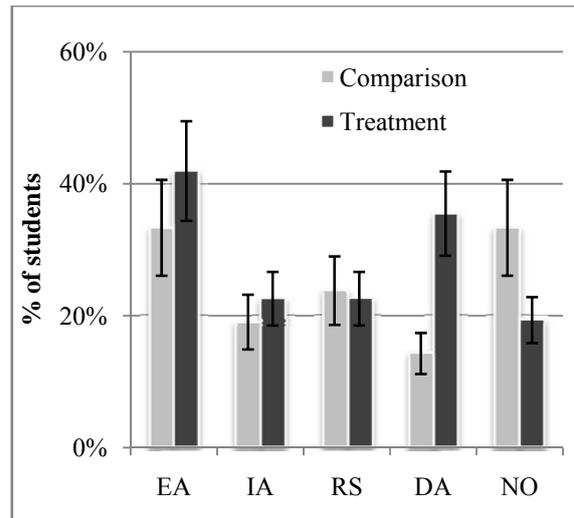
“Integral-Area relation” (IA) is the reasoning that displacement was the integral of the velocity function, and the area under the curve of velocity versus time equals this integral. This reasoning shows that students remembered the mathematical relationship between displacement and velocity function. However, it does not guarantee that students understood the relation between the integral and area under a curve concept.

“Riemann sums” (RS) reasoning indicates that students mentioned Riemann sums and related concepts, such as a left-hand rule or trapezoidal method.

“Displacement-Area” (DA) reasoning indicates that students simply wrote that the displacement is the area under the curve without providing any further justification.

“No Reasoning” (NO) indicates that no reasoning was provided by the students for their answers.

We compared the reasoning process used by two groups. The results are presented in Figure 1. In the treatment group, about 42% of the students used EA reasoning, which is an important step in understanding the area under a curve concept. But we did not see a significant difference for the two groups based on their reasoning process. Other types of reasoning, such as IA, RS, and DA, do not provide evidence of complete understanding of area under a curve.

**FIGURE 1.** Comparison of reasoning used by the two groups in response to the online hints.

Finally, the treatment group students’ responses to our final question – “How are the training problems and posttest related?” were analyzed. Students’ responses were grouped into four categories: multiplication, accumulation, approximation, and action (finding the area). The first three categories correspond to our three-step model. The last aspect – action – is indicative of student memorization of the process without evidence of conceptual understanding.

Table 2 shows an example of each category of responses while Figure 2 shows the fraction of students expressing each category of response. A majority of the students in the treatment group appear to have simply remembered the fact that they had found the area under the curve without showing any evidence of deeper understanding. However, about 20% of the students recognized the multiplication step. Several discussed how the product of the units of the two axes was physically relevant in that it yielded the units of a third quantity. About 10% of students recognized the accumulation process and 17% of students mentioned the approximation idea.

TABLE 2. Categories of responses to the question: “How are the training problems and posttest related?”

Category	Example Response
Multiplication	“You have to have some kind of variable that is constant over the interval and multiply it by Δ , whatever is on the x-axis.”
Accumulation	“They both deal with using the area under a graph to sum a variable, such as work done or distance traveled.”
Approximation	“When the ‘curve’ is rigid with corners it is easy to find the area. When it is smooth and gradual, then you must approximate.”
Action	“Both used graphs and finding the area under a curve to solve a problem.”

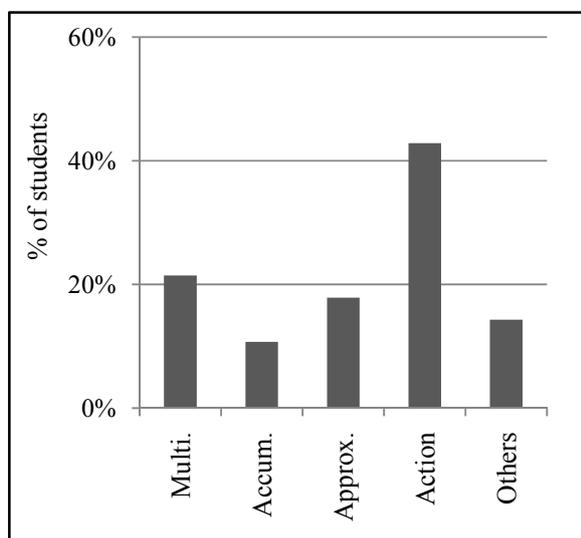


FIGURE 2. Fraction of treatment group students' responses in each category to the question in Table 2.

CONCLUSIONS

Results from the pretest and posttest task indicate that more students in the treatment group were able to use the area under the curve concept after the tutorial session than the comparison group. However, the majority of students still preferred to apply kinematics equations instead of using graphs in a kinematics situation.

From students' responses towards the online hints, we found about 42% of students in the treatment group showed an understanding of the first two layers of the three-step model by making the connection between the physics equation and area under the curve concept. We did not find a significant difference between the two groups.

Responses provided by students to the last question provided us a sense of what they think they have learned from the tutorial and online hints. Based on the data, only about half of the students appear to have acquired part of the three-step model of using the area under a curve concept. Thus, more work needs to be done to develop interventions that will successfully facilitate students to develop and apply the area under a curve concept in physics problem solving.

LIMITATIONS & FUTURE WORK

One of the limitations of our work is the short duration of the training session. Students were only exposed to our tutorial material for 50 minutes, and it is very difficult to modify or change their thinking in such a short intervention. In the future, we intend to develop tutorials teaching area under a curve on several physics topics and carry out a longitudinal study of students' learning and transfer of this concept.

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