

# Algebra-based students & vectors: can $ijk$ coaching improve arrow subtraction?

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Students in calculus- and algebra-based introductory physics courses have been shown to perform significantly better on vector addition and subtraction using  $ijk$  representation than identical tasks using an “arrows-on-a-grid” representation. Evidence supporting a knowledge hierarchy has been observed, with the ability to correctly solve  $ijk$  format questions necessary to correctly solving arrow format questions. The absence of explicit  $ijk$  instruction in typical algebra-based courses may exacerbate difficulties experienced by all physics students with vector addition and subtraction in the arrow representation in the algebra-based population. In this study we investigate to what degree instruction in the  $ijk$  format improves vector subtraction skills in the arrow format. The instruction was a one-time online intervention with feedback given to students in an algebra-based introductory physics course. While neither intervention produced gains that were statistically significant, we find evidence that students who performed well on the intervention questions perform better on a posttest question when controlling for pretest scores.

## I. INTRODUCTION

Successful manipulation of vectors and vector quantities is critical to success in introductory physics courses, and extensive literature exists examining student difficulties with vector operations, specifically addition and subtraction [1–10]. Prior work has shown that students’ vector abilities are largely unchanged even after a year of physics instruction [2], students’ solution methods are influenced by the relative position and orientation of vectors [6, 7], and that students tend to stick with the same solution method across a variety of problems, even though a different method may be more appropriate [5]. Recent work by Mikula and Heckler has labeled vector operations including addition, subtraction, dot products, cross products, and finding components as “essential skills” [12]. Of these, addition and subtraction may hold the place of “most essential,” not only due to their increased frequency (particularly in the early weeks of intro physics courses), but also because both calculus-based and algebra-based students are expected to master them, whereas topics such as dot- and cross-products are usually not used in as much depth in the algebra-based sequence.

While most of the work mentioned above has focused on vectors represented as arrows (with or without a grid), recent work has shown that students in both calculus-based and algebra-based courses perform better on vector addition and subtraction in the  $ijk$  format (when compared with arrows-on-a-grid), and that students can reason physically about their answers at least as well in either format [9–11]. This is perhaps surprising, since the  $ijk$  representation is rarely used in algebra-based courses, and is often not present in many of the corresponding textbooks (though vector components such as  $\vec{A}_x$ ,  $A_x$ , and  $\mathbf{A}_x$  are typically present) [13]. Heckler & Scaife also found evidence that a knowledge hierarchy exists: the ability to perform well on  $ijk$  problems is necessary to perform well on arrow problems [9]. The ability to use the  $ijk$  representation may allow students to reduce cognitive load by translating the vectors to a representation that allows for easier algebraic manipulation. With the extensive

evidence in the PER literature that using multiple representations is beneficial to problem solving [14], it is reasonable to ask if including  $ijk$  instruction in the algebra-based course would improve performance on other types of vector operation problems. In this study, “the  $ijk$  format” refers to vectors of any dimension written out using any combination of  $\hat{i}$ ,  $\hat{j}$ , and  $\hat{k}$  unit vectors, similar to prior work [9–11]. All questions in this study used 2-dimensional vectors.

Mikula & Heckler have recently shown that sustained, on-line, mastery-style practice on multiple types and representations of vector operations throughout a semester leads to significant performance gains on a vector skills assessment for students in a large-enrollment calculus-based introductory physics sequence, as well as increased retention of those skills [12]. This leaves open the question of whether targeted  $ijk$  instruction can help students in the algebra-based sequence perform better on vector addition and subtraction tasks, particularly in departments without the resources to implement a mastery-style online system. Here we present the results of an end-semester, one-time, online intervention designed to test if instruction in the  $ijk$  method could help algebra-based students perform better on vector subtraction questions in an “arrows-on-a-grid” context. Specifically, does instruction on translating vectors as arrows-on-a-grid into the  $ijk$  representation help algebra-based students perform better on graphical 2D vector subtraction than an intervention which reminded them of graphical vector subtraction methods they had learned during the semester?

## II. METHODS

Data were collected in the single section of a large-enrollment algebra-based mechanics course (148 students) during the spring 2018 semester. The course met for three 50-minute sessions each week with an optional 2-hour lab component. The course used a mixture of Peer Instruction & traditional methods, with regular online assignments [15]. Graphical vector addition with arrows was covered in the course as were vector components, but the specific  $ijk$  no-

What is  $\vec{A} - \vec{B}$  ( $\vec{A}$  minus  $\vec{B}$ )?

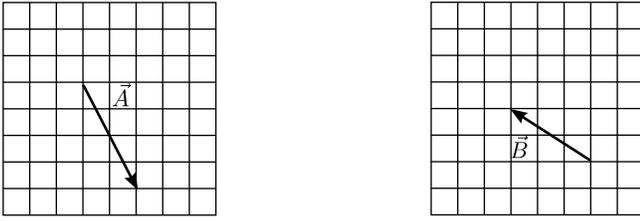


FIG. 1. Pretest (and posttest) question for both versions of the ITT.

tation was not used and vectors were not typically written out as sums of components ( $\vec{A} = \vec{A}_x + \vec{A}_y$ ). The instructor of the course did not perform any data analysis for this research.

The questionnaire, which we will refer to as the *ijk* Transfer Task (ITT), was administered online through the Qualtrics survey platform, and consisted of 8 (arrow intervention) or 10 (*ijk* intervention) multiple choice questions including feedback. The class was randomly split into two groups, one of which was given the survey with the *ijk* intervention ( $N = 74$ ), and the other the arrow intervention ( $N = 74$ ). The number of students completing the ITT in each group was 52 (*ijk*) and 54 (arrow), giving participation rate of 70% and 73%, respectively. No differences between the two groups were detected in terms of gender makeup (Fisher’s exact test,  $p = 0.42$ ) or final course grade ( $t(105) = -1.28, p = 0.20$ ). No time limit was set, but students typically took between 5 and 10 minutes to complete the ITT. A small amount of extra credit on the final exam was given to students who completed the ITT. Students were given the opportunity to opt-out of the research, either by not participating or by requesting that their responses be removed from the analysis, once final grades were posted.

The ITT consisted of a single pretest question, 6 (arrow) or 8 (*ijk*) intervention practice questions with feedback, and then a posttest question which was identical to the pretest question. All vectors that students were asked about were given in an arrows-on-a-grid format, and vectors were identical between the two versions of the ITT. Distractors were typical mistakes such as performing vector addition rather than subtraction or subtracting only one component. All distractors were identical across the two interventions, though the format differed according to the intervention type (*ijk* or arrow). The pretest (and posttest) question are shown in Fig. 1.

After completing the pretest question, each group was then given an intervention on how to correctly subtract vectors using a specific method. The intervention consisted of a worked example, chosen due to the online format and evidence that has shown worked examples can reduce the time to acquire problem-solving schema [16]. Students in the arrow group were shown how to form the negative of the second vector,  $-\vec{B}$ , then add  $\vec{A} + -\vec{B}$  using the tip-to-tail method. The *ijk* group was shown how to translate their arrow-on-a-grid into *ijk* format, subtract the two *ijk* vectors keeping track of com-

ponents, and then translate their *ijk* answer back to the grid to get the final arrow “ $\vec{A} - \vec{B}$ ”.

After the instructional portion of the intervention, both groups were given four questions to practice a key piece of their method: the arrow group was asked to identify the “negative” of a given vector (“Choose the correct  $-\vec{B}$  for the vector shown below”), while the *ijk* group was asked to choose the right *ijk* representation of the vector shown on a grid (“Choose the correct way to write out vector  $\vec{B}$  in *ijk* notation.”). After all four questions were answered, students were shown the original vector, what answer they chose, whether or not they were correct, and (if incorrect) the correct answer with an explanation for each of the four questions. We refer to these questions as “PracticeQX”, since students are practicing a smaller part of the whole.

After the four PracticeQX questions, students were given a series of questions which had them synthesize the skill they had just practiced and apply it to vector subtraction. Students in the arrow group were asked to identify  $-\vec{B}$ , and then asked to find  $\vec{A} - \vec{B}$ . Students in the *ijk* group were asked to translate  $\vec{A}$  and  $\vec{B}$  into *ijk* notation, identify the correct *ijk* representation of  $\vec{A} - \vec{B}$ , and then to choose the correct arrow representing  $\vec{A} - \vec{B}$ . Students receiving the *ijk* intervention have to translate two vectors,  $\vec{A}$  and  $\vec{B}$  from arrows into *ijk*, whereas students receiving the arrow intervention just have to find the negative of a single vector,  $-\vec{B}$ . Once the subtraction is complete, students receiving the *ijk* intervention need to translate their answer back into the arrow representation, whereas the students receiving the arrow representation would be done at this point. This results in 2 additional questions on the *ijk* intervention compared to the arrow intervention. These additional questions are necessary to determine the stage at which students have difficulty: translating the vectors to *ijk*, subtracting the *ijk* vectors, or translating from *ijk* to arrows. Students were given feedback on all steps as before. We refer to this sequence of questions as “SynthesisQX” since students were working with the same set of vectors throughout and applying the skill they had covered in the “Practice” questions. Students were then given a posttest question, which was identical to the pretest question.

### III. RESULTS

The results are shown in Fig. 2. A Wilcoxon-Pratt signed-rank test with continuity correction shows that posttest scores were not significantly different from pretest scores in either the arrow ( $Z = -0.5, p = 0.62, N = 54$ ) or *ijk* intervention ( $Z = -1.5275, p = 0.13, N = 53$ ). While the *ijk* posttest scores were slightly higher than the pretest scores, the pre- and post- scores on the arrow intervention were nearly identical. Scores on the pretest question were similar in both interventions (Wilcoxon rank sum test  $W = 1413, p = 0.89$ ).

Although there is no statistically significant difference between pre- and post- scores overall in either intervention,

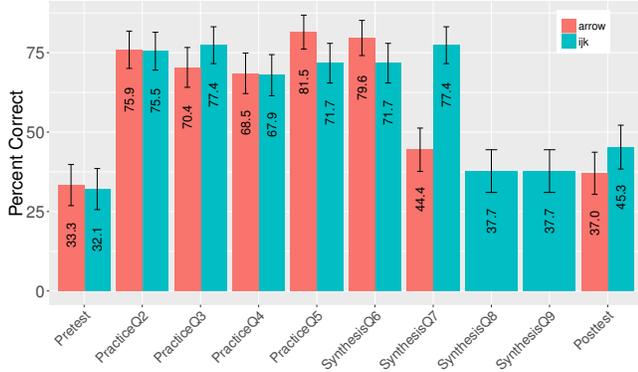


FIG. 2. Percent scores for each question on the intervention. The *ijk* intervention required two more “Synthesis” steps than the arrow, so the arrow intervention does not have an 8th or 9th question before the pretest. Error bars represent  $\pm 1$  SE.

TABLE I. Results of the logistic regression for correctly answering the posttest question. The regression coefficients ( $\beta$ ), their standard error,  $z$ -statistic,  $p$ -value, odds ratio ( $OR$ ), and 95% confidence interval (CI) for the odds ratio are listed. The \* indicates significance at the  $p = 0.05$  level.

Ind. Variable	$\beta$	$\beta$ SE	$z$	$p$	$OR$	$OR$ CI
Intervention ( <i>ijk</i> )	0.619	0.469	1.32	0.187	1.85	[0.75, 4.79]
Pretest score	0.819	0.491	1.67	0.096	2.27	[0.86, 6.03]
Component score	1.820	0.817	2.23	0.024*	6.16	[1.39, 36.13]
Subtraction score	1.510	0.493	3.06	0.002*	4.52	[1.75, 12.26]

we can still investigate how performance on the intervention might affect posttest scores, controlling for both the type of intervention and pretest score. We categorized the intervention questions into a Component score consisting of all questions where students were asked to translate an arrow into *ijk* or find the negative of a vector, and a Subtraction score which consists of the intervention questions where students were asked to subtract two vectors.

Table I shows the results of a logistic regression of the independent variables Intervention (*ijk*), Pretest score, Component score, and Subtraction score against the dependent variable of the posttest score. A logistic regression was used since our dependent variable has a binary outcome, “correct” or “incorrect”. While both the intervention type and the pretest score did not have a statistically significant effect at the  $p = 0.05$  level on the posttest outcome, their odds ratios were 1.85 and 2.27 respectively, indicating that students who had the *ijk* intervention were 1.85 times more likely to correctly answer the posttest question over those who received the arrow intervention, and students who correctly answered the pretest question were 2.27 times more likely to correctly answer the posttest question. However, since the 95% confidence intervals on both of these variables includes 1, neither was statistically significant. Both the average scores on the

Component questions and the Subtraction questions were statistically significant at the  $p < 0.05$  level, with students who answered the Component questions correctly 6.16 times more likely to answer the posttest question correctly, and students who correctly answered the Subtraction questions 4.52 times more likely to correctly answer the posttest question.

## IV. DISCUSSION

The lack of a statistically significant increase from pre- to post- for both interventions may raise concerns that students did not meaningfully engage with the ITT. To determine if students were “clicking through” to completion solely for the extra credit, we can examine the scores on the intervention questions. Figure 2 shows that scores for many of the intervention questions across both groups is substantially higher than scores for the pre- or post- questions. The “PracticeQX” questions are easier than the pre and post questions, with *ijk* questions asking students to translate a vector-on-a-grid to the correct *ijk* representation, and arrow questions asking students to identify the negative of a given arrow-on-a-grid. For the *ijk* intervention, questions “PracticeQ2-5” and “SynthesisQ6-7” were of this type. For the arrow intervention, questions “PracticeQ2-5” and “SynthesisQ6” were of this type. The higher scores in these simpler questions, 74% vs 45% posttest and 75% vs 37% posttest for *ijk* and arrows respectively, suggest that students were not randomly guessing to “click through” the intervention.

Scores did not remain at those higher levels throughout the entire intervention, however. On the last half of the “Synthesis” questions, “SynthesisQ8-9” on the *ijk* and “SynthesisQ7” on the arrow intervention, average scores dropped back down to levels similar to scores on the pre- and post-questions, shown in Fig. 2. These were the questions on the intervention where students were asked to perform vector subtraction. In the *ijk* intervention, students were asked to find the *ijk* representation of  $\vec{A} - \vec{B}$  (Q8), and then to identify the arrow-on-a-grid representation of the same  $\vec{A} - \vec{B}$  (Q9). In the arrow intervention, students were asked to identify the arrow-on-a-grid representation of  $\vec{A} - \vec{B}$  (Q7), where they had identified  $-\vec{B}$  in the previous question (Q6).

This pattern of high scores on the early intervention questions changing to lower scores on the subtraction questions could be due to several factors, each of which are hard to determine in the absence of other data (such as interviews). It may be that, regardless of the method used to subtract vectors, vector subtraction is still inherently far more difficult even when students can correctly perform the prior steps which are necessary (either correctly translating two vectors into their *ijk* representations or identifying the negative of an arrow vector). While this might make sense for the arrow representation, it would contradict earlier work we have done showing that students with no formal *ijk* training had an easier time adding and subtracting vectors given in an *ijk* format than an arrow format [10]. Another possible explana-

tion is that, consistent with work done showing that students tend to stick with a given solution method and that their solution method can depend on the orientation of the vectors [5–7], students are simply ignoring the correct  $ijk$  representations that they previously found and defaulting to a “tip-to-tail” method since the vectors are presented as arrows-on-a-grid. While the correct answer was the most frequently chosen (38% of students), the next most frequently chosen response (23%) corresponds to finding  $\vec{A} + \vec{B}$ , rather than  $\vec{A} - \vec{B}$ . This provides additional evidence that the students were not randomly guessing on the questions. While students may have simply added the  $ijk$  components rather than subtracted, our previous work indicates that this mistake is unlikely if the vectors are initially presented in the  $ijk$  representation [10, 11]. While the  $ijk$  intervention explicitly asked for the  $ijk$  representation of  $\vec{A} - \vec{B}$  and the scaffolding of the two prior questions was intended to guide students to use the  $ijk$  representations of  $\vec{A}$  and  $\vec{B}$  they had just found, we cannot be certain that the students actually did so. One possibility is that students defaulted back to finding  $\vec{A} - \vec{B}$  graphically. This is consistent with prior work [5–7, 10, 11], however direct evidence is needed before a definitive claim can be made.

The results of the logistic regression may seem redundant: students who perform well on the posttest *should* be able to answer the intervention questions well. However, this trend remains even when controlling for their score on the pretest question, students who performed better on the training questions have a better chance of answering the post-test question correctly. While a single intervention was not sufficient to raise scores overall, our results suggest that multiple interventions may be more effective.

## V. CONCLUSIONS

The primary result of this work is that a one-time, online intervention is unlikely to significantly improve the ability of students in an algebra-based introductory mechanics course

to correctly subtract vectors in a 2D arrows-on-a-grid format. This conclusion is limited in that the intervention lacked immediate feedback, did not incorporate mastery-style or interleaved learning, and only included text-and-image feedback as opposed to audio-visual. The format of the intervention, either using arrows to perform vector subtraction or first converting the arrows to the  $ijk$  representation, had no statistically significant impact on the posttest scores.

We note, however, that on both sets of intervention questions (arrow and  $ijk$ ), students performed well on the portion of the intervention that was intended to build skills needed to perform vector subtraction in either representation, which included translating an arrow-on-a-grid into the corresponding  $ijk$  format. Students’ performance on the intervention questions was a significant predictor of their success on the posttest question when controlling for intervention type and pretest score. While we do not claim causation, the fact that increased performance on the intervention questions predicted better performance on the posttest question even when controlling for the pretest score suggests that multiple interventions in either format, similar to what was done in Ref. [12], may lead to increased performance on vector subtraction problems.

Future work will investigate why students who perform well on translating arrows-on-a-grid into the  $ijk$  representation still struggle with performing vector subtraction when presented with arrows-on-a-grid. We hope to adapt the intervention presented here into an interview format so we can observe the methods students use to solve various vector addition and subtraction problems after they have completed training in either the  $ijk$  or tip-to-tail method.

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