"So it's the same equation...": A blending analysis of student reasoning with functions in kinematics

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Student use of mathematics in physics is an area of current interest in both PER and the Research in Undergraduate Math Education (RUME) community. In particular, the function concept has been widely studied in RUME but has received less attention in PER. Using a grounded approach, this study probes the ability of introductory physics students to (1) interpret graphical representations of position vs. time functions and their corresponding derivatives and to (2) translate the graphical representation into a meaningful symbolic representation. Data were collected through think-aloud interviews and analyzed using a conceptual blending framework [1]. We created what we believe to be an "ideal expert blend" of math and physics mental spaces against which the student responses were compared. We focus on case studies of two students whose approaches differed and examine how blending influenced responses to a novel graphical representation.

I. INTRODUCTION

The mathematical function is a concept that pervades much of introductory physics. In this paper, *function* is defined as a relation in which every input has a single output. Function is a subject that has been studied extensively by the Research in Undergraduate Math Education (RUME) community [2–4]. Also, much research has been done by the PER community on kinematics [5, 6], but only a fraction draws resources from, or has explicit affiliation with, the math education community. A goal of this study was to unite RUME and PER work on student understanding of the function in kinematics.

II. CONTEXT

This study was inspired by results from a set of written data [$N = 28$] collected in an introductory kinematics course in Fall 2004. Students were provided with a position vs. time graph of two objects' motions and asked to find various quantities using the graph. The last part required students to construct an expression for velocity as a function of time for one object. When answering, many students used incorrect functional notation or mixed variables such as x and t. This led to inquiry into how students view and use functions. In exploring literature on functions, we noted the similarity of this task with other research from the RUME community.

We developed a think-aloud interview protocol consisting of three parts. The first was adapted from a RUME paper about a graphical representation of a function and its derivative. The second was based on the kinematics exam consisting of finding quantities from a graph of a position vs. time function. The third part contained the same tasks as the kinematics exam, but used spacetime convention (see Fig. 1). Spacetime graphs (or Minkowski diagrams) are frequently used in special and general relativity. They contain the same information as traditional position vs. time graphs, but the orientation of the axes is switched, with position on the horizontal axis and time on the vertical.

The choice of the spacetime diagram was influenced by

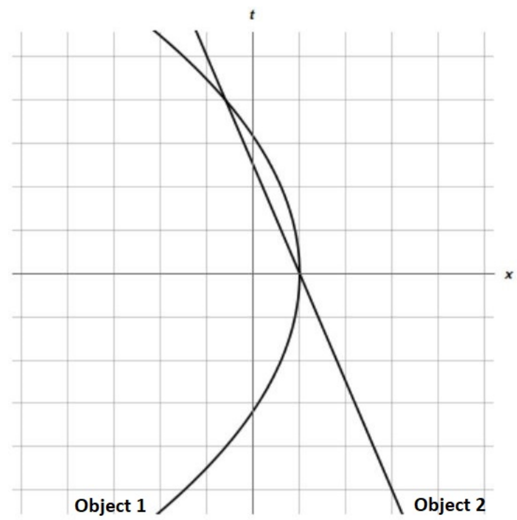

FIG. 1. Graphs for Third Task.

math education research on bidirectionality and covariation [3]. The concepts of bidirectional reasoning and covariation are necessary components in understanding concepts such as function and rate of change. RUME researchers Moore and Paoletti state that "denoting the horizontal axis as the independent/input quantity is a convention common to the teaching of mathematics," and that "students understand it as a necessary part of graphs" [3]. They suggest switching axes to determine if students are thinking bidirectionally, leading to our choice of the spacetime diagram.

III. THEORETICAL FRAMEWORK

Conceptual blending is the theory that all human knowledge is pervaded by metaphors constructed from bodily experiences; knowledge is grouped into resources that are activated together at appropriate moments [1]. In the blending framework, these resource groups are termed *mental spaces*.

When confronted with a new concept, the mind inputs mental spaces into a blend to make sense of the notion at hand [7].

Two types of blends will be considered in this paper. Single-scope blends use one of the input spaces as the framework for the blended space. Bing and Redish illustrate a single-scope blend between a math space and a physics space in which a problem is solved using math machinery, and units are tacked onto the final answer without any sense-making [8]. The math input space is the framework for the blend, and the physics is used superficially.

Double-scope blending employs input spaces to create a new mental space with elements non-native to any input space. Bing and Redish show how students used a double-scope blend of math and physics in order to conduct a physically meaningful calculation [8]. In their example, students exemplified a bidirectional flow of thought, starting in the physics space and using the math space to calculate, or beginning in the math space and using physical intuition to reason.

Single-scope and double-scope blends describe how students use math and physics mental resources to solve physics problems. In general, neither blend is more appropriate than the other, and there is a possibility of incorrect blending in either case. One of the examples we describe below shows a contrast between correct and incorrect blending approaches taken by two students on the same task.

We believe the spacetime diagram (see Fig. 1) presented an opportunity for students to create or revise a blend "on the fly". Because of its orientation, students could use previously constructed mental spaces, but would have to blend these spaces differently to accomplish tasks such as finding velocity and acceleration.

Guided by this blending framework, we thought to answer the following research questions: For students at the introductory physics level, what is the understanding of the mathematical function, and how do they relate it to concepts in physics? More specifically, how are students blending concepts from math and physics mental spaces?

IV. DATA COLLECTION AND ANALYSIS.

We performed semi-structured think-aloud interviews with volunteer students [N=7] from introductory calculus-based mechanics. All students had finished at least one semester of calculus, and came from academic majors including physics, engineering, and mathematics, but none had used spacetime diagrams before their interview. Data were collected through audio and video recordings and transcribed for analysis.

While reading the transcripts, we began qualitatively identifying the reasoning that students were using and assigning words or phrases as elements of input spaces. Initially it was assumed that only two input spaces would be used, namely a math space and a physical intuition space. After conducting a few interviews, we identified three distinct mental spaces from student reasoning: a mathematical formalism space, a graphical space, and a physical space. Based on preliminary coding, we generated what we believe to be an expert blend incorporating these spaces and the correspondence between them. In part, a well-formed blend denotes (1) the ability to connect elements of input spaces in multiple contexts, not being limited to the familiar, and (2) command over fundamental subject knowledge which aids in problem-solving.

Although prior literature did not distinguish between graphical and mathematical spaces, we chose to treat them as separate spaces. We created a graphical space distinct from a mathematical formalism space. This approach is consistent with RUME studies on covariation which is an analogous way to describe the blending that happens between the mathematical formalism space and graphical space [3].

After developing the expert blend, we looked in the interviews for specific words or phrases associated with the corresponding mental spaces. For example, *slope*, *tangent line*, or *rise over run* were categorized as elements of the graphical space. The analogous math space contained elements such as *f'(x)*, *f'(a)*, or *derivative* while the physics space consisted of expressions like *average velocity*, *instantaneous velocity*, or *change in position over change in time*. We color-coded responses to examine patterns in student thinking. In many cases, extended sequences of a single color were grouped together, indicating reasoning in a single mental space. Other situations contained multiple colors grouped together, suggesting blending. Examples of each are provided.

V. EXAMPLES OF BLENDING PHENOMENA

The blending examples below were chosen to illustrate cases in which student either students articulated elements from multiple spaces when working through a problem, or did not articulate elements of certain spaces.

A. The "Horizontal" Line Test

The first example is chosen to show the creation of a new blend. Student S1 was asked whether a position vs. time function was appropriate for the spacetime diagram. This segment portrays the adaptability of her double-scope blend.

I: Would it be appropriate to describe the position of these objects with a function, x of t?

S1: Umm... If you could like rewrite it so that like the x is, you know, there, and then t is there, then you could write it as a function, I think.

I: I will note for the audio listeners that... one of the first things you did was to actually rotate the piece of paper.

S1: Yeah... Because it can't be a function if it doesn't pass the vertical line test... if it's f of x, the f would have to be up here and then the x would be here. So if it's x of t, then the x would have to be up here, the t would have to be here.

Based on the student's responses up to this point, the student accessed all three spaces: physics, graphical, and mathematical formalism. She connected *function* and *vertical line test*, and *x of t* and *f of x*. As noted in the audio, she rotated the paper to put the axes in familiar positions, and stated that the graph did not pass the vertical line test in the original orientation, and so it would have to be rotated.

I: ...if I want x of t, I can't draw the graph like that?
S1: Um... I don't think so, no. unless... Cuz if you can't move the [axes] or like rotate it or anything, then...
S1: Let me think. I think you can.
I: You can? You changed your mind again!
S1: ...you would draw like the vertical line tests this way [draws horizontal lines], *and then it would pass it...like the vertical line test just shows...there's two inputs for the same output.*

The vertical line test was mentioned by almost all of the interviewed students. S1 articulated how the graph failed the vertical line test in its original orientation, but then decided that the test was no longer applicable to this situation. Rather than treating the vertical line test as law, she adjusted her definition to match the underlying reality of function, namely that every input has a single output, and thus created an appropriate test for the spacetime convention.

We believe that she used the mathematical formalism space blended with the graphical space to create a new mental space that did not rely solely on either input space or use either input space as the organizing frame. She would likely have more difficulty in arriving at this conclusion using a single-scope blend using the graphical space as the organizing frame.

This student's ability to blend recalls math research on bidirectional thinking. Moore and Paoletti write, "We hypothesize that students who are supported in thinking bidirectionally will construct more productive ways of thinking about function than those who are not. A student who has the opportunity to repeatedly coordinate covarying quantities... essentially establishes a relationship that entails both a function and its inverse. " [3].

B. Incorrect vs. Correct Blending Example

For this example, we compare responses to a question about the speed of two objects shown in the spacetime convention. At $t = 4$, where the lines cross in Fig. 1, the object depicted by the curved line has a greater speed. The first student had constructed a single-scope blend of the physics space and the graphical space, and makes an incorrect conclusion about the speeds.

I: At t=4, how do the speeds of the two objects compare?
S2: Object 2 is going faster than object 1.
I: And how did you know?
S2: Cuz the slope is steeper... Yeah, the linear line is going faster than object 1.

With spacetime graphs, certain automatic tendencies are now incorrect. This student stated that the slope is steeper. On a traditional x vs. t graph, a steeper slope translates to a greater speed, but in spacetime convention, this is no longer true. His blend between *greater speed* and *steeper slope* was no longer correct. He eventually remembered that the axes were switched and changed his answer, but because he was using the graphical space as his organizing frame, he interpreted a physical reality through graphical intuition.

In contrast, S1 had constructed a functional double-scope blend, utilizing a blended space to serve as the organizing frame.

I: Which object is faster?
S1: Umm... I think that object 1's speed would be faster.
I: Why is that?
S1: Because it looks like the slope is steeper.

The two answers above contain almost identical syntax for the speed comparison and follow-up reasoning, however they arrived at exactly opposite answers! Student S1 interpreted her graphical reasoning through the physical knowledge provided by the axis labels. She stated the slope is "steeper", meaning that the object was moving a greater distance in the same amount of time, even though the line is not actually steeper. The student had, in essence, redefined her graphical space through her blend, so that "steeper" now actually denoted "more gradual".

C. Initial Velocity Units Example

In this final example, a student with a single-scope blend was determining the initial velocity on the spacetime graph. She expressed discomfort while considering the units:

S3: So, the velocity of this would be the slope of this line... so -10/4? ...Rise over run...I have a feeling that this is actually supposed to be 4/10 seconds per meters, whatever that means.

The student stated that the velocity is the slope of the line, displaying an incorrect blend of the graphical and physical spaces for this problem. She was reminded of the axes:

I: Which way is seconds and which way is meters?
S3: Oh, wait hold on... this is meters, this is seconds. ...So yeah, I think it's -10/4 seconds per meters. I think that's what it is, or it's the other one, meters per seconds.
I: And then, so that's velocity?
S3: I don't know what that is.

After considering the axes, the student switched the units *seconds per meters* to match the numerical value of the slope. When prompted further to check consistency with the original task of finding velocity, she claimed to not know what her answer was. In our small data set, this was consistent with students who had constructed strict single-scope blends. Because the graphical space served as the organizing frame of the blend, the change to the normal graph rules made it difficult to use the physics space to interpret the answer.

Both this student and S2 above had excelled in answering the portions of the interview that used normal axes. Both voiced discomfort with spacetime convention, and struggled to make sense of tasks that had been accomplished previously.

D. Connecting Math and Physics

While our focus has been on the third interview task, we briefly describe the second. Its primary purpose was to document the existing blends that students had between math, physics and graphs. Our assumption was that a semester of physics would establish blends. While most students had various degrees of pre-existing blends, there were several whose responses suggested very weak connections between corresponding math and physics ideas. Students made statements suggesting that they view physics as being wholly distinct from math, and vice versa.

One student was using a traditional position vs. time graph to construct an equation for velocity as a function of time. He completed the task in a formal, mathematical way, using ordered pairs and slope-intercept form. When prompted to recall the kinematics equations, he found that he had unknowingly "re-derived" one of the fundamental equations, indicating an incomplete blend.

I: You did not [use]...kinematics equations... Would that have been relevant here?

S2: ...let's say this is velocity initial. Velocity final... Acceleration..., and time... this is v final equals acceleration times time plus velocity initial... So it's the same equation... I didn't even relate that.

VI. CONCLUSIONS

We have analyzed student responses to introductory kinematics tasks using a blending analysis. In our analysis, we characterized student use of a graphical space, which we believe to be distinct from a general mathematical input space. We found evidence of blending and used a novel representation, the spacetime diagram task, to disrupt existing blends. Students who had constructed single-scope blends, utilizing the graphical space as their organizing frame, seemed to struggle more with the spacetime graph because the information in the organizing frame is no longer presented in a recognizable way. Also, there is evidence of some students not constructing a functional blend between math and physics concepts when taking introductory calculus-based physics. The above statements may be important for instructors, educators, and researchers to note. Students form different blends which can help garner differing amounts of success depending on context, and when blends are not fully formed, elements can clash and fundamental principles may be forgotten.

This work is preliminary and involved a small number of students, but the students we interviewed came from multiple sections of the course, and a variety of math backgrounds and majors. It is possible that the particular sequence of questions primed reasoning in the graphical space, so additional work might probe this further. While we did not set out with a blending framework in mind, it seemed particularly appropriate to this set of tasks. The different approaches taken by students suggested that an epistemic games approach [9] may also be fruitful for future analysis.

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