“Rules without a Reason”: ODEs in a concept image framework

Andy Fung and Michael Loverude
Department of Physics, California State University, Fullerton,
800 N. State College Boulevard, Fullerton, CA, 92831

Elementary ODEs are seen as prerequisite knowledge gained from the introductory calculus sequence for any physics student entering the upper division. In this paper, we provide evidence that while students might be well-versed in the rules and notations of ODEs, this does not necessarily translate to the application of these “rules without a reason” to novel physics tasks. Using the mathematics education researchers Tall and Vinner’s concept image framework, we propose that the body of knowledge or concept image a student brings to an upper division physics environment regarding ODEs is restricted. We present, via four student interviews, three potential signals of this “restricted concept image”: mathematical processes that are formally taught in introductory calculus and physics courses but are not reliably evoked when faced with a novel physics task. Our goal for this paper, as part of a larger project exploring student difficulties regarding ODEs, is to create a “proof of concept” that can be used in future work to more definitively identify the presence of a restricted ODE concept image.

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I. ORDINARY DIFFERENTIAL EQUATIONS: RELATIONAL VS. INSTRUMENTAL UNDERSTANDING

According to the math education researcher Skemp[1], "understanding" can be understood as being relational or instrumental: the former refers to understanding in the *conventional* sense ("knowing both what to do and why") while the latter refers to understanding in a more *restricted* sense ("rules without a reason").These two definitions of understanding might help explain why students struggle with context-rich tasks despite showing a strong instrumental understanding in the prerequisite "rules".

Ordinary differential equations (ODEs) are used in nearly all fields of physics to model change in a system. ODEs are usually first encountered in math courses in the form of tasks that emphasize the correct application of algorithms to solve specific classes of ODEs. Advanced physics courses, however, treat ODEs as mathematical "tools": a subroutine that can be executed provided the relevant physical laws have been rewritten into a standardized form. However, recognizing that an ODE is an appropriate mathematical model for a physical system is an expression of a student's relational understanding of ODEs.

There are a few examples of previous work in PER. Black and Wittmann[2] identified several procedural resources associated with the solution method of separable ODEs. Clark conducted a survey of the state of existing ODE curriculum in math and science courses[3]. And Hyland described student difficulties with ODEs in math and physics using Tall and Vinner's concept image framework[4].

This paper adds to this body of work by suggesting that upper-division physics students' instrumental understanding of ODEs does not necessarily translate to the required relational understanding to solve novel physics problems.

To investigate, we created an interview protocol that required students to use the following mathematical skills: (1) rewriting a physical law in terms of the derivatives of a function, (2) appropriately carrying out the separable ODE algorithm, and (3) determining a physically real solution using indefinite or definite integration. All three of these processes were formally taught in previous math courses but our data shows that these skills are not easily evoked in context-rich physics tasks.

Our goal is to create a preliminary "proof of concept" in using the concept image framework to explain difficulties in ODEs students may encounter when approaching a context-rich physics tasks via the identification of potential signals that indicate the presence of a restricted concept image.

II. THEORETICAL FRAMEWORK

A concept image, as defined by the mathematics education researchers Tall and Vinner[5], is the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." In short, it is everything one knows about a particular topic.

Tasks such as exam/homework questions, interviews, and surveys can only activate or *evoke* a certain part of it (an *evoked concept image*). Difficulties arise, Tall and Vinner further explain, when this evoked concept image comes into cognitive conflict with the canonical answer, i.e., the standardized response given by a textbook or a lecturer that is expected of a student.

Framing student difficulties in ODEs as the result of a conflict between evoked concept images and pre-established theory (what Tall and Vinner call a *concept definition*) could be productive. For example, see Tall and Vinner's own example of concept images[5] of limits and continuity or Rösken and Rolka's study on student's concept images involving integration[6]. However, for our data we frame our data in terms of a *restricted concept image* or a concept image that is largely characterized by instrumental understanding: knowledge of the rules and conventions of solving of an ODE but not necessarily knowing why said rules and conventions exist.

With a restricted concept image, the student does not necessarily hold any "wrong" ideas, i.e., evoking a part of the concept image that is in direct conflict with the pre-established concept definition. However, if a task requires a student to use an ODE in a way that extended beyond a simple application of instrumental knowledge, the student will not be productive because the relational understanding needed to complete the task lies outside their restricted concept image. In other words, this restricted concept image, as summarized by Tall and Vinner, allows a "student in this position [to] operate quite happily with his restricted notion adequate in its restricted context. [They] may even have been taught to respond with the correct formal definition whilst having an inappropriate concept image. Later, when [they meet] functions defined in a broader context [they] may be unable to cope." Several studies in PER have used this notion of a restricted concept image to explain difficulties with other mathematical topics such as Fourier series (Mays)[7] and non-Cartesian volume elements (Schermerhorn)[8].

In this paper, we provide examples that suggest students approach physics tasks involving separable ODEs with an instrumental approach: an approach that was productive in previous math courses. Therefore, a **restricted concept image** is characterized primarily by a focus on instrumental understanding. Difficulties arise, however, when using ODEs in context-rich physics tasks. The necessary relational understanding lies outside of the restricted concept image.

III. RESEARCH METHODS

We conducted four interviews (identified by the labels: John, Paul, George, and Ringo). During the hour-long process, volunteers were asked to solve 2-3 tasks involving a novel physical situation. For this paper, we will be focusing on subsets of two tasks: a vertically falling particle in a medium that provided a quadratic drag force and a first-order nuclear decay. We choose separable ODEs because these are among the earliest examples of ODEs, and one that nearly all students will have encountered in math courses, even if they have not taken a course in ODEs. (For example, separable ODEs are covered in Chapter 9 of the widely-used calculus text by Stewart [9])

In previous preliminary studies using student written responses, we have observed a number of situations in which students struggled to connect the mathematical formalism taught in math to the requirements of modeling a physical situation. In math courses, the functional dependence is generally foregrounded; physics equations in contrast have a large number of additional parameters and our experience suggests that it can be difficult for students to determine which symbols represent the dependent and independent variables in a given problem. A critical part of using ODEs in physics is the use of boundary conditions and/or initial conditions; these are generally connected to the so-called arbitrary constants in an ODE solution. In practice, these often emerge when students evaluate an integral; the initial value of the velocity $v(0)$ might relate to the $+C$ in an indefinite integral or to the lower bound of a definite integral.

The volunteer pool consists of three undergraduates and one graduate student (Ringo) who received his undergraduate degree at another institution. All interviewees have taken a upper-division classical mechanics course and a "math methods course": an undergraduate course that contains the necessary mathematics for upper division physics.

For each task, the interviewer, initially, gave little additional background in addition to the given question prompt. The protocol required the student to arrive at a canonically correct solution–therefore some prompting was given after a significant period of inactivity. The interviewer also asked several questions throughout the interview asking the interviewee to clarify their written responses. For the purposes of this paper, we have selected a subset of our protocol that revealed a potential signal of a restricted concept image.

Interview Task #1 Excerpt:
Suppose a particle (initially at rest) is dropped from a tall platform in a medium that provides a speed-dependent quadratic drag force.

(a) The equation of motion is given by

$$F_{\text{net}} = mg - cv^2$$

If we wanted to find out what the velocity of the ball is at any time, i.e., find a function $v(t)$, how would we rewrite this equation?

(b) Solve the differential equation for $v(t)$ using any method.

Interview Task #2 Excerpt:

(a) The following differential equation describes the number of radioactive particles remaining in a sample that is undergoing a first-order decay.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN \quad ; \quad N(0) = N_0$$

Find a function $N(t)$ that describes the number of radioactive particles in the sample at any time.

(b) For a first-order nuclear decay, there are two ways of integrating the resulting separable ODE

$$\int_{N_0}^{N} \mathrm{d}N = \int_0^t -k\,\mathrm{d}t \qquad \int \mathrm{d}N = \int -k\,\mathrm{d}t$$

Given the auxiliary condition $N(0) = N_0$, do both of these methods of integration produce: (1) mathematically equivalent results and (2) physically equivalent results?

IV. DATA AND ANALYSIS

We present several excerpts from our four interviews as examples of three processes that are nominally taught in introductory math courses but might not be readily accessible to a student with a restricted concept image.

A. Hidden Derivatives are Actually Hidden

In ODE tasks in math courses, the function to be solve for in an ODE are generally clear. This is not the case in physics tasks–where the student must realize that a $v(t)$ term is "hidden" inside the acceleration term. A non-restricted ODE concept image would have identified that the velocity term is "hidden" inside the acceleration term to write

$$m\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right) = mg - cv^2$$

We found, however, that three out of the four students required some form of prompting to complete this task. Ringo, the sole graduate student, was able to complete the task quickly but stated that "[he was] familiar with this problem."

John was also familiar with this problem from intermediate classical mechanics but wrote an ODE whose solution was a position function $x(t)$ instead.

Interviewer: So let's imagine we want to find the velocity as a function of time, so how could we use what you have to set up an equation that would allow us to eventually find velocity as a function of time?

John: I'll just relate the second derivative to time

$$\frac{\mathrm{d}^2\ddot{x}}{\mathrm{d}t^2} = \frac{mg - cv^2}{m}$$

We note the presence of the double over-dots on the x. Because John indicated the "second derivative" during multiple attempts to write down the ODE, we believe that it is a typographical error rather than a claim that the equation of motion can be rewritten as a fourth-order ODE. After John's unproductive attempt to solve for $x(t)$, the interviewer redirected John to the task prompt (to find $v(t)$).

Interviewer: So, ultimately, we want v as a function of t not x as a function of t.

John: Right.

Interviewer: So how could you do this by thinking about v as a function of t. Does that make sense?

John: Uh, I believe so? So, we would just isolate that v and moving everything over to the left side?

Interviewer: Again, remember the prompt: "to relate speed to its derivatives". So are there any quantities here that are related to v.

John: Yeah, the acceleration? So...

John then wrote

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{mg - cv^2}{m}$$

This selection of $x(t)$ by John suggested to us that information from physics (the relationships between the functions of position, velocity, and acceleration) did not automatically transfer to the mathematical language of the ODE–which requires a function $v(t)$ to be useful since there is already a term of v on the right hand side. It is only after prompting by the interviewer that John was able to write the ODE in its expected form. Interestingly, John did rearrange the equation in a way that isolated the derivative term–using such language such as "isolating" and "moving over". These operations suggest that John is familiar with the notation and algebraic rules of an ODE (a sign of instrumental knowledge). These algebraic rearrangements, however, do nothing to engage with the hidden derivative.

A similar conversation developed for the remaining two interviewees. Paul first isolated the v term on the right hand side(an expression of instrumental understanding) –writing

$$v = \sqrt{\frac{mg - ma}{c}}$$

and only realized that there was a hidden velocity derivative hidden inside the a term after being prompted by the interviewer.

George pursued a similar strategy of isolating the acceleration term [but left out a mass term].

$$a = mg - cv^2$$

When asked by the interviewer how acceleration is related to velocity function $v(t)$, George responded correctly that the acceleration was the derivative of the speed function but was unsure about how that information would change his ODE. This is apt evidence of a restricted concept image: George instrumentally recited that the derivative of $v(t)$ gives the acceleration but could not relate this understanding to the task in the form of an ODE.

George: We don't have any t variables in here. We don't really know how this all moves in time.

Interviewer: So, if we replaced the a with, like, $\mathrm{d}v\,/\,\mathrm{d}t$. Would that help?

George: Yeah, that would help.

John, Paul, and George had the instrumental knowledge to isolate the unknown variable v and even recognize that it was related to the acceleration. However, they were not productive in this specific task because they lacked the deeper relational understanding of how all of these factors come together in the form of an ODE: something that does not exist in a restricted concept image.

B. Mathematically Equivalent Integrals Are Not Seen as Physically Equivalent

Integration is a critical part of the solution process for many ODEs, particularly the separable equations described in this paper. Subtask (b) of the second task involved answering a question about the differences between definite and indefinite integration. Both methods contain all of the information needed to provide an accurate physical model. Several students struggled with this, not because they were unable to execute the integral correctly but rather because they had difficulty choosing whether to use definite vs. indefinite integrals.

For example, Ringo stated that while both solutions led to similar looking results, the two results were not equivalent because of the arbitrary constant that resulted from the indefinite case. The arbitrary constant is "[something] we're going to use eventually, like, if we were doing experimental physics. Like, if we knew the constant, then we would get the result." Ringo then went on to clarify that indefinite integrals were mostly used in "first and second year physics, but that in the higher levels or when we were doing experiments, we would use definite integrals."

It is unclear from this interview which form of integration Ringo finds most useful for physics. However, it is noteworthy that he believes that there is a difference. All of the physical information is present for both methods of integration. This, combined with the statement that one method was used significantly more than the other over the course of his physics education, suggests that while he is instrumentally aware of the rules of both methods of integration, he might not be relationally aware that all of the necessary information needed to construct a realistic physical model is present in both integrals.

Paul made similar statements when asked the same question of mathematical and physical equivalence

Paul: Probably not mathematically equivalent because you're giving yourself bounds–like if you had this small number to this big number for the bounds while this one (gestures to the indefinite case) it's harder to work with because you don't know where you're going to start.

Interviewer: So, going to the physically equivalent part...

Paul: I feel like the actual answer you would get is from this top part (gestures to the definite case) because it'll tell you physically that there's a start and an end. You have to account for that. While (gestures to indefinite case), you're gonna get a really general case.

Paul's claim that the two methods of integration are not physically equivalent (the definite case being more accurate as it accounts for the start and end) is similar to Ringo's views.

Paul and Ringo's responses suggest to us that the ability to see indefinite and definite integration as equally valid methods of generating a physically real model (the choice depending on mathematical convenience) is not present in their restricted concept image for this particular task. In math courses, the decision to use definite or indefinite integrals is clear: definite integrals will have bounds attached to them in the problem statement and indefinite integrals require a $+C$ term after calculating the anti-derivative. For many tasks in physics, however, students are required to choose an appropriate "branch" of of a procedure/algorithm such as whether or not to use a definite or indefinite integral. The inability to do so indicates to us that the criteria for choosing, a demonstration of relational understanding, are not well-understood for the student.

C. Separating ODEs that Are Not Separable

The first two cases strongly suggest the value of the restricted concept image. The following example is not as clear as it could have been interpreted as a procedural error. This suggests that the restricted concept image may incompletely describe our data set. For the following excerpt, we again note that the double over-dots on the position differential seem to be a minor mathematical copy error.

John: In order to solve for this, we have to...Basically what mathematicians love is to probably split this

$$\frac{mg - cv^2}{m}\mathrm{d}^2\ddot{x} = \mathrm{d}t^2$$

Interviewer: So, what would you do next based on what you have here?

John: So, we just integrate both sides with whatever differential is respect to it. [After integrating, note the procedural error]

$$\left(g - \frac{2cx}{m}\right)\mathrm{d}\dot{x} = \mathrm{d}t$$

Hmm, I'm not sure. Just kinda on autopilot right now.

Interviewer: You were a little bit hesitant. Can you say more?

John: More in the sense in that I'm making the correct relationships between velocity, acceleration, and position. And how they relate to each other in terms of derivatives of each other but I don't know how to translate that into differential form.

We note that the decision to use the algorithm for separable ODEs was done spontaneously (or to borrow John's vocabulary: "on autopilot"). It is only after when he integrates the second order derivatives that he stops. This response may be interpreted as a signal towards a restricted concept image. It is also possible that he would've performed the same procedural error regardless of whether or not his concept image was restricted.

V. CONCLUSION

In this paper, we have presented multiple pieces of data that we interpreted as evidence of a restricted concept image. As stated previously, a restricted concept image is characterized by an emphasis in *instrumental understanding*. In the context of ODEs, having a relational understanding of ODEs allows the student to recognize an ODE resembles a canonical form that has a solution algorithm, e.g., the separable ODE algorithm. Tasks that involve relational understanding, such as being able to rewrite a physical law in the form of an ODE, will lie outside of this restricted concept image.

For example, we have shown multiple instances of students being unable to proceed because the problem does not have the surface features of the standard procedure or algorithm. This was the case for three out of the four students in the first interview task. Prompting was needed in order for the student to rewrite Newton's second law in the form of an ODE. Furthermore, we classified the tendency to continue to execute steps of a procedure/algorithm when they are not/no longer appropriate (as John did with the separable ODE algorithm) as another signal of a restricted concept image. Finally, we interpret the inability to choose an appropriate "branch" of a procedure/algorithm (as Paul and Ringo did with the decision to use an indefinite integral versus a definite one) as a signal of a restricted concept image.

We stress issues such as algebraic errors (such as the one John made with the $\mathrm{d}^2\ddot{x}$ term are not evidence of a restricted concept image). Furthermore, any procedural errors/difficulties, e.g., difficulties with evaluating an integral, are not interpreted as evidence of a restricted concept image.

While the grain-size of the concept image framework is quite large, its usage to interpret our data set suggests that many problems in which students model physical phenomena are likely to be at or outside the bounds of their individual ODE concept image. Therefore, it will not be uncommon for students to possess a restricted concept image when entering a physics course, despite having completed a course in all of the mathematical prerequisites.

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