# Development, validation and online and in-person implementation of clicker question sequence on quantum measurement uncertainty

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Research-validated clicker questions comprise an easy-to-implement instructional tool that can scaffold student learning while formatively assessing students' knowledge. We present findings from the development, validation and implementation, in consecutive years, of a Clicker Question Sequence (CQS) on measurement uncertainty as it applies to two-state quantum systems. This study was conducted in an advanced undergraduate quantum mechanics course, in both an online and in-person learning environment. Student learning was first assessed after receiving traditional lecture-based instruction on relevant concepts, and their performance on it was compared with that on a similar assessment given after engaging with the CQS. We analyze and discuss similar and differing trends observed in the two modes of instruction.

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## I. INTRODUCTION

In quantum mechanics (QM), one type of fundamental measurement uncertainty occurs when an observable is measured in a state that is not an eigenstate of the corresponding operator. The measurement collapses the state to one of the eigenstates of that operator with a certain probability, and the measurement uncertainty is the standard deviation of many measurements of this observable conducted in an ensemble of identically-prepared systems. Furthermore, any subsequent measurements of the same observable made in a collapsed state, assuming no time-evolution, will yield the same outcome with 100% certainty. An observable is said to be well-defined in an eigenstate of the corresponding operator. Also, the uncertainty principle states that if two observables have corresponding Hermitian operators that do not commute, then they cannot both be measured with 100% certainty, and thus cannot both be well-defined, in the same quantum state. It is important for students to recognize when observables can be well-defined in the same state by determining whether the corresponding operators commute. Because the uncertainty principle can be challenging for students, instructional resources have been developed to help students learn this concept in different situations [1]. Such fundamental tenets of quantum theory as measurement uncertainty and the uncertainty principle are relevant in many contexts, including those involving successive measurements of different observables, e.g., in the actively growing field of quantum information science.

Prior research suggests that students in QM courses often struggle with many common difficulties [2–18]. However, research-validated learning tools can help students develop a robust knowledge structure [19–25]. For example, our group has developed, validated and implemented Quantum Interactive Learning Tutorials (QuILTs) with encouraging results on many topics in QM [1,26–32]. Other commonly used learning tools in physics include conceptual multiple-choice questions known as clicker questions. In the method first popularized by Mazur's *Peer Instruction*, these questions are presented for students to answer anonymously, first individually and again after peer discussion, with immediate feedback [33]. They have proven effective and are relatively easy to incorporate without greatly restructuring a typical course [34]. When presented in validated Clicker Question Sequences (CQS) on a topic, they can systematically help students who are struggling with particular concepts. Previously, such CQSs related to several key QM concepts have been developed, validated and implemented [35–39]. Furthermore, previous work has been conducted to investigate student difficulties with the uncertainty principle as it applies to wavefunctions [40] as well as two-state systems [41], but there has not yet been a documented effort to develop, validate and implement a CQS to address those difficulties. Here we describe the development, validation, and implementation of a CQS intended to help students learn about measurement uncertainty as it pertains to two-state quantum systems, and we discuss difficulties in identifying well-defined observables in a given state, calculating measurement uncertainty, successive measurements of various spin angular momentum observables, and other difficulties that naturally came up during implementation.

## II. METHODOLOGY

The developed, validated and implemented CQS is intended for use in upper-level introductory QM courses. The data presented here are from administration in a mandatory first-semester junior-/senior-level QM course at a large research university in the United States. During the development and validation process, we took inspiration from a previously-validated QuILT and CQS on the uncertainty principle for wavefunctions—determining learning objectives and adapting some questions—as much research involving cognitive task analysis has been conducted from both student and expert perspectives [1]. We also drafted and iterated new questions for uncertainty related to two-state systems. To ensure that the material could be completed in the allotted class time, while offering maximal value to students, we prioritized the coverage of conceptual knowledge, used common difficulties as a guide, and provided checkpoints that could stimulate useful class discussions. We iterated the questions many times amongst ourselves and with other faculty members to minimize unintended interpretations and ensure consistency and simplicity in terminologies and sentence constructions.

The 17 questions in the final version of the CQS focused on the following four learning goals: identifying well-defined observables in a given quantum state (4 questions), calculations related to measurement uncertainty and the generalized uncertainty principle (5 questions), commutation of $\hat{S}^2$ with $\hat{S}_z, \hat{S}_x$, and $\hat{S}_y$ (5 questions), and compatible vs. incompatible operators (3 questions).

Selected CQS questions, referenced in later sections, are reproduced below with answers in boldface. The states associated with the *z*-component of spin with eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are $|+z\rangle$ and $|-z\rangle$, respectively (similar notation for the *x*- and *y*-components). All other notations are standard.

**CQS 1.1** Consider a system with a Hamiltonian $\hat{H} = C\hat{S}_x$, where $C$ is an appropriate constant. Choose all of the following statements that are correct for the system if it is in an eigenstate of $\hat{S}_z$, i.e., $|\pm z\rangle$.

I. The observable $S_x$ is well-defined.
II. The observable $S_z$ is well-defined.
III. Energy is well-defined.

A. I only **B. II only** C. II and III only
D. I and III only E. All of the above

**CQS 2.2** Choose all of the following statements that are correct about the uncertainty in the measurement of an observable $A$ in state $|\chi\rangle$ (which is not an eigenstate of $\hat{A}$).

I. $\sigma_A^2 = \langle(A - \langle A\rangle)\rangle^2$

II. $\sigma_A^2 = \langle A^2\rangle - \langle A\rangle^2$

III. $\sigma_A^2 = \langle(A - \langle A\rangle)^2\rangle$

A. II only B. I and III only C. I and II only

**D. II and III only** E. All of the above

**CQS 2.5** The generalized uncertainty principle for two observables $A$ and $B$ is ${\sigma_A}^2{\sigma_B}^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$. For $S_x$ and $S_y$, ${\sigma_{S_x}}^2{\sigma_{S_y}}^2 \geq \left(\frac{1}{2i}\langle[\hat{S}_x,\hat{S}_y]\rangle\right)^2$. Choose all of the following statements that are correct for a system consisting of a spin-1/2 particle.

I. If the state of the system is $|+z\rangle$, we have $\langle[\hat{S}_x,\hat{S}_y]\rangle = \langle+z|[\hat{S}_x,\hat{S}_y]|+z\rangle = 0$.

II. If the state of the system is $|+y\rangle$, we have $\langle[\hat{S}_x,\hat{S}_y]\rangle = \langle+y|[\hat{S}_x,\hat{S}_y]|+y\rangle = 0$.

III. Measuring the observables $S_x$ or $S_y$ in the state $|+y\rangle$ will yield $\frac{\hbar}{2}$ in each case with 100% certainty.

**A. II only** B. I and II only C. II and III only

D. All of the above E. None of the above

**CQS 3.5** The operators $\hat{S}_x$ and $\hat{S}^2$ commute with each other. Suppose you measure $S_x$ in some initial state, for a spin-1/2 system, and obtain a value of $\frac{\hbar}{2}$. What will the state be if $S^2$ is measured in immediate succession?

**A. $|+x\rangle$** B. Either $|+x\rangle$ or $|-x\rangle$

C. The system can be in any eigenstate of $\hat{S}_x$, $\hat{S}_y$, or $\hat{S}_z$

D. The system can be in any state in the Hilbert space after the measurement, since every state is an eigenstate of $\hat{S}^2$.

E. Not enough information

**CQS 4.3** Choose all of the following statements that are correct if $\hat{A}$ and $\hat{B}$ are *incompatible* operators with non-degenerate eigenstates.

I. It is impossible to find a complete set of simultaneous eigenstates for $\hat{A}$ and $\hat{B}$.

II. In a given quantum state, for three successive measurements $A \rightarrow B \rightarrow A$ (*assuming no time evolution of the state has taken place*), the two measurements of A must yield the same value.

III. It is possible to infer the value of the observable $B$ after the measurement of the observable $A$ returns a particular value for $A$.

**A. I only** B. III only **C.** I and II only

D. All of the above E. None of the above

The final version of the CQS was implemented once online and once in person. The CQS during the online implementation was presented as a Zoom poll with questions displayed via the "Share Screen" function, and for the in-person implementation the poll was replaced by a functionally similar classroom clicker system. For each question, after displaying the polling results, the instructor held a full class discussion of the possible options provided. The Peer Instruction component was present in the in-person administration, but not the online administration because of difficulties in fostering small-group student discussion in the online environment. We note also that the instructors were different for the online and in-person classes.

To determine the effectiveness of the CQS, we developed and validated a pre- and post-test containing questions on topics covered in the CQS. The post-test contained small changes such as a shift from eigenstates of $\hat{S}_x$ to eigenstates of $\hat{S}_y$, but otherwise remained conceptually similar. In both online and in-person classes, students completed the pre-test immediately following traditional lecture-based instruction on the topic. After administration of the CQS over two to three 50-minute class sessions, students completed the post-test. While students may be primed by the pre-test to pay more attention to the CQS, the mode of instruction was familiar to the students, as it had been used to teach several concepts. Two researchers graded the pre-test and post-test, and converged after discussion on a rubric inter-rater reliability greater than 95%. Selected pre- and post-test questions are reproduced below; some other questions are not examined here due to limited space. Questions Q1 and Q2 on the pre- and post-test provided students three possible answers from which to choose, and credit of up to 3 was awarded for correctly selecting or omitting each answer. (For these two questions, correct answers are bolded.) For the free-response questions, Q3b and Q4a-d were scored with 2 points, one each for answer and reasoning. A more detailed breakdown of the questions is provided in Section III.

**Q1.** Consider a system in the state $|+x\rangle$. Choose all of the following observables that are well-defined in this state (i.e., they can be measured with no uncertainty):

**I. $S^2$** **II. $S_x$** III. $S_z$

**Q2.** Consider a system in the state $|+y\rangle$. Choose all of the following observables for which measurements would yield a value with 100% certainty:

**I. Energy, if the Hamiltonian is $\hat{H} = C\hat{S}_z$ (where $C$ is an appropriate constant)**

**II. Energy, if the Hamiltonian is $\hat{H} = C\hat{S}_y$ (where $C$ is an appropriate constant)**

**III. Any observable whose corresponding operator commutes with $\hat{S}_y$**

**Q3.** For the following states of a system, does a measurement of the observable $S_x$ yield a value with 100% probability? If the uncertainty is non-zero, calculate it.

**a.** $|+x\rangle$

**b.** $\frac{1}{3}|+x\rangle + \sqrt{\frac{8}{9}}|-x\rangle$

**Q4.** Consider the Hermitian operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}^2$. Suppose you first made a measurement of the observable $S_x$ for a system in some state, and obtained the value $-\frac{\hbar}{2}$. For each of the following cases, what are the possible values that you obtain for the final measurement, and what is the state immediately after the final measurement?

**a.** You immediately make another measurement of $S_x$.

**b.** You immediately make a measurement of $S_y$.

**c.** You measure $S_y$ in immediate succession, and then $S_x$ once again.

**d.** You measure $S^2$ in immediate succession, and then $S_x$ once again.

## III. IMPLEMENTATION RESULTS

The pre-test and post-test results, as well as normalized gain [42] and effect sizes [43], are listed in Tables I (online with $N = 27$) and II (in-person with $N = 23$). Overall, the results are encouraging, and students performed well on the post-test, with relatively high normalized gains and generally medium to large effect sizes. Below, we investigate some difficulties that were addressed during the administration of the CQS for both years, as well as one that remained for smaller percentages of students.

TABLE I. Results of the online administration of the CQS. Comparison of pre- and post-test scores (rounded), along with normalized gain (*g*) [42] and effect size as measured by Cohen's *d* [43], for students who engaged with the CQS ($N = 27$).

| Q# | Pre-test | Post-test | *g* | *d* |
|---|---|---|---|---|
| **Q1** | 78% | 94% | 0.72 | 0.93 |
| **Q2** | 94% | 98% | 0.60 | 0.33 |
| **Q3a** | 78% | 85% | 0.33 | 0.19 |
| **Q3b** | 52% | 76% | 0.50 | 0.71 |
| **Q4a** | 87% | 93% | 0.43 | 0.21 |
| **Q4b** | 69% | 81% | 0.41 | 0.39 |
| **Q4c** | 70% | 83% | 0.44 | 0.38 |
| **Q4d** | 37% | 76% | 0.62 | 0.91 |

TABLE II. Results of the in-person administration of the CQS. Comparison of pre- and post-test scores (rounded), along with normalized gain (*g*) and effect size as measured by Cohen's *d*, for students who engaged with the CQS ($N = 23$).

| Q# | Pre-test | Post-test | *g* | *d* |
|---|---|---|---|---|
| **Q1** | 80% | 99% | 0.93 | 1.06 |
| **Q2** | 91% | 97% | 0.69 | 0.32 |
| **Q3a** | 65% | 96% | 0.89 | 0.88 |
| **Q3b** | 35% | 78% | 0.66 | 1.09 |
| **Q4a** | 65% | 96% | 0.88 | 0.99 |
| **Q4b** | 52% | 85% | 0.68 | 0.85 |
| **Q4c** | 46% | 80% | 0.64 | 0.82 |
| **Q4d** | 48% | 70% | 0.42 | 0.51 |

### A. Difficulties that decreased significantly after CQS

#### *1. Identifying observables that are well-defined in a state*

Questions Q1-Q2 on the pre- and post-test asked students to identify observables that are well-defined in a given state. Most students correctly selected $S_x$ or $S_y$ identifying that the state is an eigenstate of the corresponding operator. In Q1, some students did not select $S^2$, which is also well-defined in the given state because its corresponding operator is proportional to the identity operator and commutes with $\hat{S}_x$. Some students incorrectly selected $S_z$, which may be due to the use of $S_z$ in class as a frequent example particularly for $\hat{H}$. On the CQS, questions such as CQS 1.1 addressed these issues. On the post-test, the full correctness rate across all three answers on both Q1 and Q2 was increased, with high effect size for Q1 (see Tables I-II).

#### *2. Calculating the measurement uncertainty*

On the pre- and post-test, Q3 presented students with two states (Q3a and Q3b). The question asked them to determine whether the uncertainty in measuring a particular component of spin was zero in the given state, and to calculate the uncertainty if not. Across both implementations, we gave full credit to students who were able to provide the formulas for calculating uncertainty (e.g., an observable $A$ has uncertainty $\sigma_A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ and the symbols under square root are the expectation values of $A^2$ or $A$). With the primary learning goal assessed by the question being the problem-solving approach, our rubric avoided penalizing students who were unable to produce the correct calculation due to a mathematical error. Also, some students correctly identified whether the measurement uncertainty is zero or non-zero, but justified their answer only by invoking the probabilities of measuring each possible outcome. These students received half credit. Questions such as CQS 2.2 address issues related to the calculation of measurement uncertainty, and while Q3a had a large effect size in the in-person implementation, there were reasonably impressive normalized gains and effect sizes for Q3b across both years (see Tables I-II).

#### *3. Results of successive measurements of $S_x$ and $S_y$*

On the post-test, Q4 asked students for the final outcome of consecutive measurements of some permutation of the observables $S_x, S_y$, and $S^2$, specifically testing whether students could recognize what happens when the measurements involved in the question corresponded to operators that did or did not commute with each other. Question Q4a asked about two consecutive measurements of $S_x$, Q4b about consecutive measurements of $S_x$ and $S_y$, and Q4c about consecutive measurements of $S_x, S_y$ and $S_x$ again. For Q4a, most students correctly answered that the measurement in the collapsed state will yield $-\frac{\hbar}{2}$ as the outcome, but some students did not recognize that $\hat{S}_x$ and $\hat{S}_y$ do not commute. Thus, for question Q4b, they answered that the outcome of the $S_y$ measurement would still be $-\frac{\hbar}{2}$, corresponding to $|-y\rangle$, and that in Q4c the final $S_x$ measurement would yield $-\frac{\hbar}{2}$, corresponding to $|-x\rangle$, neither of which is correct. Questions such as CQS 2.5 addressed these issues, and in general, the post-test scores show reasonable improvement for all parts of Q4 during the online and in-person administrations (see Tables I-II).

#### *4. Results of successive measurements of $S_x$ and $S^2$*

Question Q4d asked students for the final outcome of consecutive measurements of $S_x, S^2$ and $S_x$ again. On the pre-test, some students stated that for the final measurement of $S_x$, either eigenvalue $\pm\frac{\hbar}{2}$ and eigenstate $|\pm x\rangle$ could be obtained, and some explicitly cited $\hat{S}_x$ and $\hat{S}^2$ not commuting with each other for their reasoning. CQS 3.5 addresses measurement of $S^2$ immediately after $S_x$, and CQS 4.3 and other questions in that sequence helped students generalize from spin-1/2 systems to more generic observables that correspond to operators that do or do not commute, and how such relationships may affect the measurements of those observables in a given quantum state. Student post-test performance showed better understanding of these concepts.

#### *5. Conflation of eigenvalues and eigenstates*

In the in-person implementation, on some questions only on the pre-test (not the post-test), some students wrote, e.g., that "The state will collapse into $-\frac{\hbar}{2}$," or "It is in the state $-\frac{\hbar}{2}$." While the CQS did not explicitly focus on distinguishing between the collapsed state and measured value of the observable, it is likely that the precise language used throughout the CQS helped students distinguish between eigenvalues and eigenstates on the post-test.

### B. A difficulty that was less successfully addressed: Interpretation of measurement

Across both years' implementations, there were some students who on both the pre- and post-test answered that the result of a measurement was the eigenvalue multiplied by the eigenstate. For question Q4 specifically, they appeared to think that making a measurement of $S_x$ in the state $|-x\rangle$ would yield an outcome of $-\frac{\hbar}{2}|-x\rangle$. They thus claimed, e.g., that the outcome for Q4a would be $\frac{\hbar^2}{4}|-x\rangle$, the outcome for Q4b would be $\pm\frac{\hbar^2}{4}|\pm y\rangle$, and the outcome for Q4c outcome would be $\pm\frac{\hbar^3}{8}|\pm x\rangle$. This type of reasoning may be closely related to the student difficulty that an operator's action on a quantum state represents a measurement of the corresponding observable in the state [40].

## IV. COMPARISONS BETWEEN ONLINE AND IN-PERSON IMPLEMENTATIONS

It appears that average pre-test performance in the in-person implementation was, in general, somewhat lower, but post-test performance for both groups was comparable. For free-response questions, the CQS helped both groups, and the normalized gains and effect sizes tended to be higher for the in-person implementation due to the lower pre-test scores.

It is interesting that students performed about equally well on the post-test for both administrations, given that the online learning environment had greatly reduced opportunity for Peer Instruction. We acknowledge that one possibility is students' ability to consult resources, despite being instructed not to do so, during the online-administered pre- and post-tests. Even though they were told that the quizzes were closed-book and closed-notes, there was no way to enforce such a rule for students who had their cameras off. Even so, those students would have had access to the same resources during both the pre- and post-test, so the sizable improvements in the post-test scores of the online class are still a good sign of the benefits of the CQS. On a separate but possibly related note, there were many instances where multiple pre-test questions were left blank during the in-person administration, but not the online administration, which contributed to the noticeably lower pre-test scores for the in-person administration. This may be indicative of some form of test anxiety that did not appear during the online year, or of the phenomenon described above. Finally, as we have noted before, the different instructors between years could have also been a factor; e.g., it is possible that, during the online administration, more emphasis was placed on content related to two-state spin systems. In summary, the administration of the CQS in an online learning context may have affected student performance differently as compared to the in-person administration, but both classes have clearly benefited from the CQS regardless.

## V. SUMMARY

Validated clicker question sequences can be effective tools when implemented alongside classroom lectures. We developed, validated, and found encouraging results from implementation of a CQS on the topic of measurement uncertainty in two-state quantum systems. Post-test scores improved for every question following the administration of the CQS. While the performance on the multiple-choice questions was high to begin with on the pre-test, there was significant improvement in the free-response questions. Effect sizes varied for the online implementation, but notably were large for nearly every free-response question in the in-person implementation due to lower pre-test scores.

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