A pilot study on draft tutorials for the metric in general relativity

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The metric tensor is a key concept in general relativity, necessary to describe black holes, gravitational waves, the expanding universe, and other interesting and important phenomena. Unfortunately, it is usually presented either very superficially or in a highly technical and mathematical way. I have begun creating tutorials to help students develop a deeper understanding of the metric and a greater facility with its use, focusing on upper-division undergraduate students. Analysis of student behavior when working with drafts of the tutorials, combined with post-instruction interviews and testing with a novel metric, suggest that these tutorials show promise to be effective for introducing students to general relativity. In this preliminary study the tutorials increased conceptual understanding, developed mathematical facility, and instilled the attitude that general relativity is an approachable subject.

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I. INTRODUCTION

Concepts from general relativity (GR) capture the imaginations of the public and students at every level. We cover black holes and expansionary cosmologies everywhere from from public talks and general education astronomy courses to modern physics courses, upper-division astrophysics and cosmology courses, and graduate courses in physics and astrophysics. Gravitational waves are now routinely detected [1], and GPS technology would not function without taking GR into account.

Thus students are fascinated by the topics GR addresses, and can encounter them in many courses other than those dedicated solely to GR. But students—and let's be honest, some instructors—can be intimidated by the math behind GR. As Einstein himself said of Minkowski's geometric union of space and time, it "makes rather great demands on the reader in its mathematical aspects" [2]. And that was before it was curved. Physicists rarely receive any training in differential geometry, except those few who enroll in GR classes. As a result it is easy to be overwhelmed by the mathematical demands of covariant derivatives and Christoffel symbols, and lose track of physical insight.

There have been strides in recent years to expand undergraduate education in GR [3, 4], and in teaching special and even general relativity to secondary school and lower undergraduate level students [5, 6]. But the gap between rubber sheet analogies and the tensor equations of GR remains considerable, and incompletely filled.

At the upper undergraduate and beginning graduate levels, texts needing results from relativity do often discuss the metric. It is key: this rank-2 tensor describes the curvature of spacetime and the coordinate system used. It is a necessary quantity to calculate proper distances or times between spacetime events, to calculate the action along spacetime paths, and it even determines causality [7, pp. 71, 161].

In many cases this taste of GR is approached via the line element, which avoids explicit tensor notation by relating infinitesimal distances to infinitesimal changes in coordinates, such as the familiar $dl^2 = dx^2 + dy^2 + dz^2$ in flat 3-D space with Cartesian coordinates. Common metrics presented include the Schwarzschild metric [8, p. 97], which describes spacetime around a spherically symmetric mass (including a nonrotating black hole), and the Friedmann–Lemaître–Robertson–Walker (FLRW) metric [9, p. 36], which describes a homogenous expanding (or contracting) universe.

However, these presentations often are cursory, leaving students without prior background confused, or they are quite mathematical (e.g., [7]). And rarely do even astrophysicists see the Kerr metric, which describes spacetime around rotating black holes. Astrophysical black holes often have high rotation rates [10], and thus the Kerr description is more accurate than the Schwarzschild description, but its higher level of complication often leads it to be omitted entirely from discussion. At the other end, the Minkowski metric describes flat spacetime, the limit that GR turns into special relativity (SR). However, many presentations of SR omit the metric entirely [11–13], focusing on aspects like the Lorentz transformations, and leaving students ill prepared for GR courses.

A key tool that has been developed for increasing student understanding is the tutorial, worksheets that structure student dialog in a setting with an instructor present to guide. They have long been successfully used at the introductory level in both physics and astronomy [14–18]. The tutorial approach has also been taken up in some areas of upper-division physics, such as quantum mechanics [19, 20].

I wanted to test if tutorials could be effective in GR as well. In this paper I start to address that question with draft tutorials focused on the metric tensor. While not ready for wide deployment, these tutorials show promise for this topic that arises in undergraduate courses on astrophysics and cosmology and at the end of courses on SR, in addition to dedicated GR courses. The metric allows students to find interesting results related to black holes and cosmology without additional mathematical complication [8]. It also lays the groundwork for students to deal with tensors more generally in the manner needed for both the Einstein and geodesic equations [21].

II. TUTORIAL DESCRIPTIONS

The four tutorials were designed to each introduce one key concept, building on previous tutorials, and ending with the full metric of GR. Unlike tutorials in other areas of physics, they were designed to be the primary instructional instrument on this topic. I used lecture minimally, and spontaneously in response to issues that appeared to need more instruction.

Tutorial 1 reviews the ideas of bases and components in familiar flat two-dimensional space. It then motivates the need for a nontrivial metric even here, when a non-orthogonal coordinate system is used. This introduces the metric without the extra cognitive load from relativity.

Tutorial 2 moves into the Minkowski spacetime of SR, asking students to apply the metric concept of tutorial 1 to explain four-vectors and dual vectors, and inner products in flat spacetime. Here I take the opposite approach of tutorial 1—the coordinate system is orthonormal, but the spacetime is not Euclidean. So the metric is simple but it is not the identity the way it is in flat *space*, because the sign must change between the spatial and temporal components.

Tutorial 3 moves to spherical coordinates, which are crucial for analysis of most physical problems. The tutorial first employs these coordinates in 3-D Euclidean space like tutorial 1, allowing students to see the coordinate-dependent nature of the metric, since they are describing the same space they would with Cartesian coordinates but finding a metric that is not the identity, even though the coordinate system is orthonormal. The tutorial then returns to the flat spacetime of tutorial 2, and finds the metric for this in spherical coordinates, $ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2$.

Tutorial 4 finally employs the metric in curved spacetime, focusing on the Schwarzschild metric, using spherical coordinates like tutorial 3. The calculation of proper time in curved spacetime is introduced, and students use this to find the event horizon of a Schwarzschild black hole. They also calculate clock drift for GPS satellites relative to clocks on Earth.

III. METHODOLOGY

In fall 2023 I taught a relativity course at Eastern Washington University, an inclusive, medium-sized, public primarily undergraduate institution offering a physics bachelor's degree. While the relativity course had previously been exclusively special relativity, I chose to dedicate the last three weeks of the term to tutorials in general relativity.

The pilot study described here includes the deployment of the draft tutorials in a real class, combined with post-instruction testing and interviews. The first goal was to establish plausible efficacy of the tutorials in building student skill with the metric, and the second goal was to examine student attitudes towards the topic and mode of instruction. Third, I planned to substantially improve the tutorials based on student interaction and observation of misconceptions.

A. Sample selection

While upper-division class sizes are typically small at this university, this course was smaller than usual, with only two students. I will use the pseudonyms Chris and Andrew. Both students agreed to participate in the study, though no incentives were offered, and it was made clear that their participation was voluntary and unrelated to their course grade.

For assessing the effectiveness of the tutorials, I used classroom observation of the students' work, their written work on the tutorials and their final exam, interviews with each student separately, and a novel test tutorial I asked them to work on together at the conclusion of the course. IRB approval was obtained to conduct and record the interviews, as well as to use student artifacts.

For the interviews I chose a semi-structured design to elicit student feedback on their attitudes and perceptions of the tutorials and their effectiveness. I wanted some feedback on common topics from both Chris and Andrew, but I wanted enough flexibility to follow anything interesting that arose. I interviewed each student separately, recorded the interviews, and then transcribed them. I also brought them together to have them work together on some problems related to the Kerr metric, which was novel and poses some complications relative to the earlier metrics they had seen. Due to scheduling limitations during finals week, Chris completed this tutorial after his interview, but Andrew completed it before.

It would certainly be preferable for the instructor to *not* also be the interviewer and the tutorial creator. Interviews would also ideally be coded by multiple people independently, to assess inter-rater reliability. However, this preliminary study was unfunded, and therefore of limited scope. Follow up work will require the participation of multiple institutions, to increase sample size, minimize bias, and allow for control groups.

IV. RESULTS

Tutorial 1 was intended to quickly recap necessary topics from linear algebra, and also introduce the concept of the metric in a nonorthogonal coordinate system. In classroom deployment it became obvious that these two goals required division into two separate tutorials, and the first tutorial should go more in depth on the necessary topics of linear algebra. Chris had not previously taken linear algebra, while Andrew had, but neither had much familiarity with the concept of bases. (Neither had yet taken quantum mechanics, where this concept is most fully developed in the standard physics curriculum.) Linear algebra courses, particularly those focused on applications for engineers and computer scientists, may focus heavily on topics like systems of equations and give short shrift to the concept of basis vectors. At the outset, neither Chris nor Andrew could define "orthonormal."

However, Tutorial 1 was effective in getting the students to rethink topics they thought were familiar, engaging with the metric in an environment detached from relativity. Given a vector, both students easily completed the familiar task of finding its magnitude. When the coordinate system was changed to keep the x-axis but switch to a y' axis halfway between the x- and y-axes, both students did eventually find the components in this new basis, but this required considerable thought and discussion. In their discussion, they noted "the length must be the same" because "the vector hasn't changed at all," which helped guide them away from naive notions about the components. Interestingly, although the topic had nothing to do with the physics of relativity, both students drew on their recent experience with special relativity as an analogy to what was happening. They related the changing components to observers finding different distances and durations between events when they work in different reference frames.

Seeing the expected consistency of vector magnitude when using a new, non-identity metric, and the lack of consistency when simply adding the components in quadrature, convinced the students that the metric was a necessary part of taking an inner product. Near the end of tutorial 1, Chris made the crucial observation to Andrew, "maybe every time you've dotted vectors you've used this [implicitly]."

Tutorial 2 was mostly review for Chris and Andrew, since we had covered vectors, dual vectors, and how to change between them in our discussion of special relativity. It was intended for practice, but also to help them realize that our specific prescription for going from a vector to a dual vector was in fact just a special case, and that $\eta_{\mu\nu}$ was just one particular example of a metric $g_{\mu\nu}$.

One issue that students must grapple with is that there is no standard choice of metric signature. Some authors choose the time component to be negative, while others choose the space components to be negative; physics only demands that their signs be *opposite*. Tutorial 2 has students calculate whether two particular events are timelike or spacelike separated, repeating the calculation in both metric signatures to verify that they agree on this observable. Neither student had difficulty with this, because the definitions were laid out in the tutorials and the notion of metric signature had been introduced earlier in the SR part of the course. Both were able to articulate clear definitions of timelike and spacelike separated that did not depend on the metric signature. For example, Andrew wrote that "timelike separated means they are within each others light cone or reverse light cone," while Chris noted that spacelike separated events "cannot be causally linked."

For tutorial 3, although both students were familiar with spherical coordinates, neither had used them extensively (both being near the beginning of the upper-division sequence). This caused some issues, such as when Andrew forgot to square the terms when writing out the length, initially giving $dl = dr + rd\theta + r\sin\theta d\varphi$. An improvement to the tutorial would be to explicitly call back to the Cartesian case, in which squaring is second nature for students at this level.

In the spirit of helping student fluency in dealing with different notation, tutorial 4 set $c = 1$ in some places, and kept c in other places. (G was retained in all situations, as is common in GR textbooks.) This was not unfamiliar to the students, who had been calculating the spacetime interval (implicitly using the metric) throughout the SR part of the course, in which we initially included c factors but eventually used $c = 1$ everywhere. However, the students did find the inclusion of the c factors helpful in making sense of the Schwarzschild radius derived in this tutorial, and while they were comfortable working with the metric both ways, when including the c factors they both forgot that it was no longer the case that $ds = d\tau$. One way of solving this would be to let $c = 1$ everywhere, but to give a value for G in these units.

One aspect of tutorial 4 that needed refinement was addressing up front the coordinate singularity of the Schwarzschild metric when using the common Schwarzschild coordinates. When asked to consider a spaceship approaching an object in a spacetime with this metric, Chris immediately considered cases of $r < 2GM/c^2$. The statement on the first page that this metric works only "outside" the object did not explain what that meant, nor what a coordinate singularity is for black holes, the one kind of object where "outside" can include such small radii.

Tutorial 4 had the students calculate time dilation for GPS satellites using the Schwarzschild metric. While both students thought black holes were interesting, they expressed great satisfaction with this problem that, in their words, was "real world" and "not an extreme situation." It is worth recalling that with undergraduate courses on relativity we're mostly not training those who will work on gravitational theory—some are going into experiment, some are going straight to the private sector, and some, like Chris, are primarily engineering majors for whom physics is a second degree.

The students had a comprehensive final exam, so most of it was on SR, the main topic of the course. However, one question had them calculate time dilation in a Schwarzschild metric, but this time in the case of a particle orbiting at the innermost stable circular orbit of a nonrotating black hole. Both students did well on this problem, showing calculational facility with the topics of tutorial 4.

In post-instruction interviews, both students showed a reasonable understanding of the metric. For example, Chris defined the metric by saying "it relates how different axes interact with each other." He related this back to the first tutorial, saying "how does x interact with y, and how does x interact with y'."

However, there were some lingering misconceptions. While working on the tutorials, Andrew sometimes included coordinate infinitesimals dX^μ (e.g., dr) in the metric tensor itself. And although Chris did not seem to have this same issue when working the tutorials themselves, it did show up in his post-instruction interview:

Interviewer: "Does velocity show up in the metric itself?... In this one frame I have this metric, then I look at another frame that's moving relative to that one, does it have a different metric?"

Chris: "Yes. So it does show up there. Just in if there's a $d\theta$ or a $d\varphi$ over time, that is a velocity. I'd say one metric that didn't have that motion, that whole component would be zero. Whereas this other frame that does have a $d\theta$ or a $d\varphi$ over time, it's going to have something there."

When asked to consider the Minkowski metric of SR, Chris correctly reasoned that "the metric wouldn't have changed" when considering an object at rest in one frame versus an object moving in that frame. This prompted him to change his mind about the general case.

Interviewer: "You're certainly getting at something when you describe this here. Is dr nonzero or not? That changes how we do things. But does dr show up in the metric?"

Chris: "Yeah. . . the metric would be almost invariant, regardless of which frame you're in."

Chris then expressed the opinion that this SR example of metric invariance would be helpful to include in the tutorials, to help clarify the difference between $g_{\mu\nu}$ and dX^ν.

Chris and Andrew then sat together to work on a new tutorial that covered the Kerr metric (in Boyer–Lindquist coordinates). This could be used in regular instruction, but here I used it as a post-instruction test to see how well students could adapt what they learned to a new situation. Kerr was a novel metric for these students, and a "mess" [7, p. 261] intended to elicit exactly the reaction that Andrew expressed when pointing to the off-diagonal elements in the matrix representation of the metric:

Andrew: "Well, my first thought is, what do we do with these guys up here on the sides? In the other metrics they were just diagonal. That was pretty much the line element, the line down the middle."

Chris: "There we have to go back to what our metric summation looks like."

Andrew: "Oh, yeah."

The cognitive load when learning relativity was evident when Andrew became distracted by the way Chris wrote squaring a vector (the "metric summation" he was referring to, seen in the upper left in Fig. 1). Andrew got confused by L^μ, which he associated with the Lorentz transformation matrix. While Chris had remembered the correct formula, and he knew how to interpret the symbols he was using, he was not using the X^μ notation that we had used in class. However, after discussion both Chris and Andrew understood the approach and wrote the correct line element (Fig. 1).

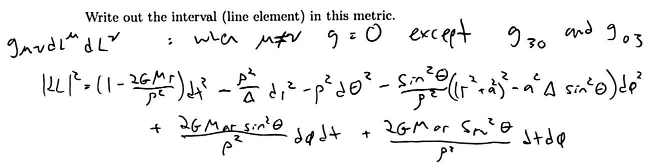

FIG. 1. The Kerr line element created from the matrix representation, from Chris's Kerr metric tutorial, with some nonstandard notation.

Over the course of this Kerr tutorial, Chris and Andrew were able to construct an expression for the proper time experienced by an object in a circular orbit around the axis of symmetry in the Kerr metric, shown in Fig. 2. Working together was important—initially Andrew omitted the $dt\, d\varphi$ term and Chris made an algebraic error.

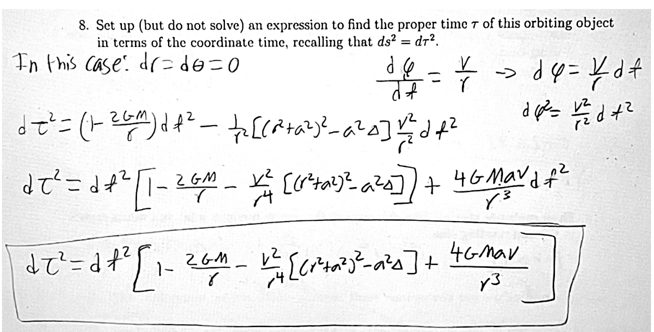

FIG. 2. Proper time for a circular orbit in the Kerr metric, from Andrew's Kerr metric tutorial, with an initial omission of the $dt\, d\varphi$ term.

V. DISCUSSION

Overall the draft tutorials show promise for instilling both understanding of and mathematical facility with the metric tensor. There are numerous areas that need refinement, and some extensions or modifications are necessary to counteract persistent misconceptions, but this has been a far more successful engagement with the metric than I have seen in my astrophysics class (where I use Maoz's well-written but terse text [8]).

Interviewer: "Do you remember in Astrophysics [a course Andrew had taken just six months earlier] when we did a little bit with the Schwarzschild metric?"

Andrew: "I remember that vaguely. That was difficult. I remember it not being easy."

The tutorials were also effective in giving the students the attitude that they could do GR and use the metric. They appreciated the insights into earlier work. As Chris said, "just the fact that that's been the backbone of everything I've done in the past in normal Euclidean space was pretty crazy to learn."

Andrew opined that "the packets do a good job of taking the stuff we learned throughout the course and just building baby steps on that into general relativity. It makes it much more approachable when you slowly creep up to the metric in spherical, curved spacetime, other metrics like the Schwarzschild and Kerr metric, rather than just 'here's the Schwarzschild metric in curved spacetime in spherical coordinates, good luck!' "

The success of tutorials does depend on the willingness of students to engage with them. While neither Chris nor Andrew had used formal tutorials in a class before, it is important to note that my pedagogical approach in the SR portions of the class focused heavily on the students themselves constructing results. This prepared them for a tutorial approach, and the present tutorials would themselves prepare the students for a worksheet approach to some more advanced topics [22].

VI. CONCLUSION

The implementation, testing, and interviews show that the tutorials I have drafted demonstrate promise to help students develop an understanding of the metric tensor. However, the tutorials are still early in the development stage. More work needs to be done to gather data on student responses to traditional pedagogical approaches to the subject, including common misconceptions. Additional analysis must be done to understand how students interact with this material depending on institution type, student demographics, and course focus. (Undergraduate relativity classes do not all cover the same material the same way, with some treating the spacetime interval as fundamental and others overlooking it entirely.) Development of highly refined tutorials in physics and astronomy has taken decades of work by many people. I have only just begun this work, but I believe that within a year or two draft versions of these tutorials could find use in the classroom. Like doctors, teachers must act with the tools and knowledge of the time, and cannot wait to act until the research is completely settled.

Another possible direction for the tutorials is adaptation to astrophysics or cosmology courses that want to truly emphasize the use of the metric. Since these are not dedicated relativity courses, it will likely be necessary to create a condensed version of the tutorials that focuses on important elements for these courses (such as proper distance and proper time), while omitting the strong emphasis on elements that are less necessary.

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