One student's resources throughout a triple integral question

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Students in physics courses are often asked to compute integrals that are both similar and different compared to the integrals from Calculus courses. We interviewed three students at a mid-sized midwestern university, and asked them to work on integral problems from math and physics contexts and talk through their thinking. We identified five conceptual resources that students activated throughout and across the interviews. Here, we highlight two of the five resources through an example of a student reasoning through a volume integral problem, and their thought process as they attempt to connect different conceptions of integration. We argue that this case study excerpt is representative of some of the hurdles that certain conceptions of integrals may lead to when solving different integral problems in physics despite students' prior success in math courses.

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I. INTRODUCTION

Mathematics content knowledge and techniques are essential tools in physics. Current curriculum standards require that students take 4-5 courses in Math departments (often Calc I, II, III, Linear Algebra and some course(s) on Differential Equations) to prepare students for using these mathematical tools throughout their physics course sequence. Previous work in PER and Research in Undergraduate Mathematics Education (RUME) literature has begun to cast light on subtle differences between the practices of doing math in math classes and doing math in physics classes [1–4]. Math and Physics have distinct disciplinary cultures, with their own shared values and practices [5]. Because of this, a confounding factor in studying this issue is that physicists who attempt to write math problems will often encode physics elements into problems even if they don't set out to explicitly include any physics context. Similarly, mathematicians often write physics questions that won't look like authentic physics problems to most physicists. For instance, physicists may write math problems with notations that do not explicitly identify the assumptions being made. Similarly, mathematicians may write physics questions that lack units which are commonly used by physicists, or ask questions that are not relevant to the physical context [1].

Literature in PER and RUME have investigated students' understanding of integration [4, 6–11]. However, investigations have yet to use integration questions written by a mathematician and physics questions written by a physicist to explore the subtle differences in students' resource activation across these differing contexts. To this end, we designed an interview protocol that uses math questions from a paper written by Jones [4] a member of RUME, but substitutes that study's "physics" questions for more context-authentic questions written by a physicist (Author WC). Our research question is: How is student resource activation impacted by integration questions written by a mathematician vs. a physicist?

II. LITERATURE REVIEW

Researchers in PER have been investigating how students understand different features of integration in physics contexts. Students may view differentials as "little bit" of a variable, connected to physical significance or as 'punctuation' that only signifies what variable is being integrated [12]. Hu and Rebello categorized students' resources concerning the differentials in integrals such as dx, dl, etc. in the context of electrostatics problems [6]. Meredith and Marrongelle noted that students' entry-level understanding of integrals as based on limits might relate to the inability to view integrals as sums [7]. There are several ways that students may interpret the definite and indefinite integral, which is reflected in how they verbalize their conceptions when encountering integral problems [8].

It is important for us to better understand the resources that students are activating when working on integrals within physics problems to aid students in navigating barriers to better understand conceptual ideas. For instance, Wallace and Chasteen noted that students had difficulty with Ampere's law because they did activate resources involving the conception of integration as a sum [10]. In thermodynamics, students may have difficulties interpreting P-V diagrams because of a lack of experience with integrals that are path dependent but share endpoints [11]. Additionally, students who maintained the same conception of an integral in a math context throughout different situations may not be as successful in problem solving or may not hold cohesive understandings [9].

A. Resources and Framing

For this study, we employ the Resources and Framing Theoretical Framework [13]. Resources and Framing models thinking as consisting of fine-grain "nuggets" that a person has learned and used successfully somewhere previously that are unconsciously brought to bear on a novel task (Hammer, private communication). These resources may have connections to other resources some of which may be stronger or weaker connections depending on the experience of the person who has used them together in the past [13]. The accessibility of these resources are influenced by how the individual frames the particular task. For example, a problem that seems very mathematical might not cue resources pertaining to physical reasoning, even if they might be productive. Therefore, just because someone doesn't articulate a particular resource, does not mean that they don't possess that resource, rather, the resource was not activated. Within this framework, we as researchers attempt to identify the "good idea" that the individual is attempting to use for a particular problem and if it is successful (i.e., achieves a correct answer), we deem that resource to be productive [14]. If the individual is not successful (i.e., gets an incorrect or incomplete answer) we deem that resource to be unproductive. The length restrictions of this paper limit our ability to go into full detail of the conceptual and procedural resources identified in our data set; however, a future paper will significantly elaborate on our findings.

III. METHODS

Participants for the study were recruited from a second-semester calculus-based physics course towards the end of the semester. Participants responded to a series of questions in a one-on-one 1 to 1.25 hr long interview using a semi-structured, think-aloud protocol [15]. A sketch or equation was written on a whiteboard by the interviewer ("INT", Author WC) and the participants were given oral prompts on how to proceed with the problem. Participants then wrote out their work on the board and were asked to describe their thinking out loud. Interviews were audio and video recorded.

Pictures were taken of the participants' completed work using a handheld camera before the board was erased to proceed to the next question. Auto-generated transcriptions were made by uploading the audio of the interview to the auto-transcription feature of the online version of Microsoft Word. These transcripts were then edited for accuracy while re-watching the video to form the final transcript.

The research team viewed the videos through the lens of a resources and framing theoretical framework [13]. Our protocol builds upon Jones (2013) [4], by adapting its interview questions for use in a different context. While Jones' work focused on identifying the symbolic forms that students used, wrote, or commented on, our study was designed to capture the cognitive resources involved in students' thought processes through their writing, gestures, and responses to verbal follow-up questions. We also differ in what resource ideas we focus on, based on the statements and actions of the students in our study. For instance, we identified resources similar to what Jones calls the "adding up pieces symbolic form", and the "the differential as a shape symbolic form" with terminology that is more reflective on the focus on "area" in our interview questions and student utterances. Our essential goal was to look for good ideas, from an expert's perspective, that the participants brought to bear on the questions/prompts regardless of whether they were productive in that situation or not, and if these varied between math and physics contexts.

The case study presented here is of Student 1 ("S1"). The final physics questions of the interview protocol are of interest due to the combination of resources present in S1's out-loud thought process and the problem-solving steps they took. A future paper will provide a complete documentation of the resources we identified from all three interviews.

A. Interview Protocol

The interview protocol was adapted from a series of interview questions written and administered by Jones, a mathematician and Mathematics Education Researcher [4]. Jones wrote a series of six math questions and six physics questions to investigate student reasoning on integration, focusing on the use of the Symbolic Forms frameworks. However, the authors of this paper had concerns with some features of the physics questions. We deem that these questions are not truly representative of typical physics questions a physicist would construct or ask. Hence, new questions that are more typical of questions a physicist would ask were generated to probe student reasoning of integration in authentic physics contexts. These questions were intended to not require extensive physics content knowledge to complete them. Instead, the goal was to gain insight into student reasoning on integration within a physics context/framing without requiring sophisticated physics understanding.

IV. ANALYSIS

A. Case study: Student 1 interview excerpts

PhysQ2 The integral of dm: The next prompt asked interviewees to think about the integral of dm, with m being the mass of an object. In most physics textbooks, this type of integral is presented when calculating the center of mass of an object. This is conceptually challenging, as the writing of bounds on this integral is physically confusing since there isn't an initial mass and a final mass, and the mass isn't changing. The implicit step is that the solver integrates over the domain of the object.

Without additional information, S1 wrote down "$m + c$" as an answer to this question, which is not what the interviewer expected. There is a way of interpreting this as correct, if we assume the m refers to some amount of mass and the constant accounts for any additional mass that would constitute the entire mass of the object. In physics, the integral of dm is often interpreted to be the total mass of the object (wherein we make an assumption that the integral is over the entire domain of the object). However, none of the three interviewees interpreted the integral in this way.

We identify this move as a procedural resource for integrating polynomials. The same type of procedure one would execute if faced with the mathematical problem $\int dx$. An evaluation of the anti-derivative would give you $x + c$. As such, there's little evidence that S1 was involving any physical insight in this problem.

The way this problem is typically solved is by using the relationship between density as the mass to volume (or area or length) ratio, which holds for the total mass and total volume but also for the ratio of $dm/dV = \rho$. Physicists may conceptualize this as "a little bit of mass (dm) divided by a little bit of volume (dV)". The relationship can be re-written as $\rho \cdot dV = dm$ and this can be substituted into the equation to complete the integration with identified spatial bounds.

PhysQ3&4 Area of rectangle and triangle: In the next question of the interview, the interviewer drew a picture of a rectangle on an x-y axes (see Fig.1), and asked if interviewees could "find the area of the box using calculus" (The triangle is added as a second prompted question). While the reader might not see this as a problem with deep physics context, it does involve using mathematics to represent a physical situation. We assert that the question is more representative of an integration question that would be asked (or done) in a physics setting, as opposed to a contextless function given in a math setting. S1 immediately said, "This is a constant function... The function that we are going to be integrating is b...", and proceeded to solve the problem completely correctly, writing the initial integral as $\int_0^a b\,dx$.

The interviewer then added the triangle and asked S1 to find the area of this combined shape using calculus. S1 stated that "We can just keep this same initial part (pointing to the integral he started with), and then we'll tack on this (points to triangle region) and add (writing an integral) of this func-

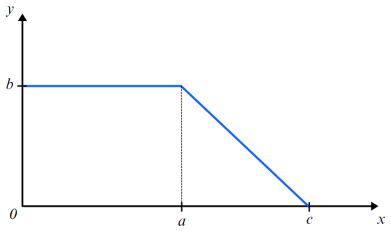

FIG. 1. The rectangle area prompt, with the triangle that was added in the second part of the prompt (PhysQ3 and PhysQ4)

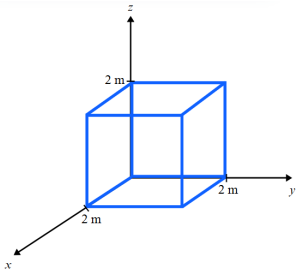

FIG. 2. The mass of a cube prompt (PhysQ5)

tion (referring to the added diagonal line) (pauses to think)." S1 assigned the function $f(x)$ to the line that forms the blue line of the triangle, and correctly wrote down the appropriate additional integral that would give the total area.

The interviewer then asked "what, if anything, does this expression mean to you?" and wrote down $\int_0^b a\,dy$ on the board. S1 again correctly interpreted the integral that would also determine the area of the box, integrating with respect to dy with ease. We identify this to be "Area as $\int$ of function" resource. Knowing that the area can be determined by integrating the function that defines the curve.

The interviewer than asked if S1 "could evaluate the area of the rectangle using a double integral?" S1 had no issue writing down the double integral of $\int_0^a \int_0^b dx\,dy$.

PhysQ5 Mass of a cube: S1 is asked to find the total mass of a cube of wood using calculus. The problem is provided on the board, with values of 2 meters in each coordinate direction (see Fig.2), and a density of 100 kg/m^3: "Can you use calculus to determine the total mass of the object?"

S1 references the previous problem that involved a double integral (and used the "Area as $\int$ of function" resource), and he immediately writes down $\int\int\int_D \rho\,dx\,dy\,dz$ where the D denotes the integration will be over the relevant domain. S1 articulates that ρ is constant, so he pulls it out of the integral, writes down the bounds for each integral, and solves the problem, adding the appropriate units at the end. We could identify this as the "Area as $\int$ of function" resource, although here it would be the multivariable equivalent. In this case, however, it seems as if the student simply knew how to do this problem.

After solving the triple integral and getting a result, INT asks S1 if they had solved for the total mass:

S1: So I know this ($\int\int\int_D \rho\,dx\,dy\,dz$) is mass because this integral ($\int\int\int_D dx\,dy\,dz$) will solve for the volume of the entire cube. And then when you multiply the volume by the density, which is in kilograms per meter cubed, meter cubed will cancel out meter cubed and you will end up getting uh Kilograms ...

INT: How do you know $\int\int\int_D \rho\,dx\,dy\,dz$ is equal to total mass? I think you may have already said that, but if you could...

S1: OK so. So density is volume or is is mass per unit volume and with the triple integral. It's similar to like when you take integral of like acceleration or velocity.

S1: ...So, like acceleration is in meters per second squared, we take the integral of that. It's basically like you're multiplying by seconds, so it will become like meters per second, and then it'll become meters when you integrate velocity again. Because you're like with the the small change like the D– like the 'dt's or whatever like acceleration, 'dt's are essentially your unit of time times acceleration will get you a velocity kind of thing.

S1: Here: Similar idea for this (points at $\int\int\int_D \rho\,dx\,dy\,dz$) like it's 100 kilograms per meter cubed. When you do the first one, it'll become 100 kilograms per meter squared. Next 100 kilograms per meter and then next one will be 100 kilograms. So like all the meters will be canceling...

After this, INT gives this follow up prompt after writing $\int\int\int_D \rho\,dm$: "Can you at all relate that expression to what you wrote before?" After a couple pauses to think, and clarifying the written form of the integral to $\int\int\int_D dm$, S1 comments

S1: So we're saying this triple integral dm is the same thing as this ($\int\int\int_D \rho\,dx\,dy\,dz$). We have to say that dm is equal to $\rho\,dx\,dy\,dz$...

As S1 is thinking, he holds onto the idea that dm represents a change in mass, and is unsure of what the bounds of integration will be: "From zero to..."

S1: dm is our our change in mass, and we're going to be integrating over three variables, so. Uh.

INT: I might actually be confusing this. Sorry, this is just I I'm I'm also kind of [confused] here– I want to get rid of this ($\int\int\int_D dm$ into $\int_D dm$) its probably more fair to do this.

Here, we could comment on how the interviewer's conceptions of the integration have been reflected on their initial question notation, and that misalignment with the students conception of the integral adds to the barriers to solving this problem. In a more common example in physics, one could write the integral over a whole volume as $\int dV$ After choosing a coordinate system, they may then write $dx\,dy\,dz$. Does this mean that writing dV would be wrong? Is this an in-between step that might support or hinder reasoning? The answer to that question depends on the resource that is being used to conceptualize the integral in that moment.

From the "Integral as adding up tiny pieces" (or "Area as integrand * d[ ]" as the resource we identify from our data)

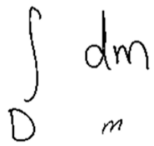

FIG. 3. S1's final integral: Integral of dm with a "D" denoting domain, and the result of that integral, m

perspective, the "$\int$" integral symbol suggests that you will add up infinitesimal pieces of a certain variable. In a physics context, this is implied to be over the bounds/domain of the object at hand (in this instance, over the entire cube). However, from the perspective of the "Integral as a mathematical operator/dx as punctuation" resource, the triple integral with dm would be seen as incorrect, as there are too many "$\int$" integral symbols for the one dm at the end. This might be analogous to having a sin function with three parentheses at the front: $\sin(((x)$

Using this perspective, we could hypothesize that S1 is using the "Integral as a mathematical operator/dx as punctuation" resource here, or a related resource. Seeing a notation that does not support the way they are thinking of the written expression could be a factor in what is hindering their understanding. When the notation is changed to reflect their current conception of integration, they are able to make further progress on the problem:

S1: Yeah, so something like with the previous one ($\int\int\int_D \rho\, dx\, dy\, dz$), or you're like– the function over it's domain kinda thing.

S1: So the domain of this ($\int dm$) includes this entire volume right here (hovering over the cube drawing) (see Fig.2).

S1: So integrating over the entire domain times the change in mass will end up giving you. The total mass (gestures holding an object), because it's– that's like what we said with the other function. It was like the function over summing... like that function I guess.

S1: So... Yes. So this ($\int dm$) will end up giving us: Integral dm over its domain (see Fig.3)

At this point, INT re-establishes the question at hand: whether there is a way to connect the integral from the first part of the problem ($\int\int\int_D \rho\, dx\, dy\, dz$) to the integral of mass, now written as ($\int_D dm$). The interviewer believed this would be an ah-ha-type moment since S1 had so quickly and easily written out the triple integral with ρ, dx, dy, and dz, but it took considerable contemplation time. S1 eventually makes the connection between dm and dV

S1: So let's say. dm/dV... (pause) Changing mass respect to volume. Is equal to density I guess.

S1: ...So dm is equal to $\rho\, dV$. In this case (pointing at cube) the change in volume. So we could rewrite this as integral over its domain of dm. Is equal to the integral over the domain of $\rho\, dV$

S1: And then $\rho\, dV$ – or dV would just be $dx\, dy\, dz$

S1: So we can do the triple integral. of $\rho\, dx\, dy\, dz$.

S1 does write down the appropriate relationship, but doesn't conceive of dm and dV being little bits of mass and Volume, and instead reads the relationship as changing mass with respect to volume. This is similar language to how this expression is commonly read in instances like $\frac{dx}{dt}$, as the change in position with respect to time.

As a debrief to this question, which was the last formal question in the interview protocol, INT provides some context to what S1 has accomplished in this problem:

INT: It's something that you mentioned earlier on when you were describing the like, what the integral was and you drew that little tiny slice of stuff. In three dimensions it's a little bit trickier, but really that dm represents a little tiny piece of mass or you could– That's the way that physicists often like to think about it, mathematicians get a little wary.

INT: But if you think about this as a little tiny piece of mass, then absolutely right. There is a ratio of any little type of density [thats] uniform. The density is equal to the total mass divided by the total volume, right?

This excerpt serves as a reflective summary of the differences reasoning (and resources) that may be in play for problems in math or physics contexts, like the problems in the interview, through the eyes of an expert researcher that is conducting research on the boundary of PER and RUME.

V. CONCLUSIONS

Investigating subtle differences between math and physics practice may identify previously less well understood barriers for students learning physics. This analysis of this brief scenario demonstrates someone who is very knowledgeable about integration. This individual is successful in nearly all problems presented previously, but struggles to make the connection between dm and spatial integral of $\rho\ dx * dy * dz$. Thus, we have presented a brief scenario of someone who earlier demonstrated solid knowledge of integration, who struggled to make an explicit connection between a mass integral and a spatial integral in a physical context. Our analysis, albeit limited within the page requirement, shows that this individual did not activate the most productive resources for this situation, and rather used resources that are better suited for situations in mathematics contexts. This demonstrates that even under ideal circumstances, students may still struggle to interpret infinitesimal dm (or other differentials) as little bits of mass, an interpretation that physicists often use and is essential for productively assembling and understanding integrals in many physics courses.

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