Student Understanding of Superposition: Vectors, Waves, and Wave Functions

Paul J. Emigh
Gina Passante
Peter S. Shaffer

Physics Education Group (Research, Curriculum, Instruction)
UNIVERSITY of WASHINGTON

What is superposition?

Superposition is a physical principle that allows two (or more) examples of the same physical quantity to be added. Often, the *direction* of the quantities is necessary, in addition to the magnitude.

Why study superposition?

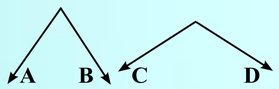
Superposition is everywhere in physics. It is one of the earliest topics studied in mechanics, and it is necessary for phenomena like motion, water waves, electromagnetism, and quantum mechanics.

How do we study superposition?

We asked students in introductory and upper-division physics courses to compare the resulting magnitudes when two quantities are superposed in different ways. We gave a wide variety of different physical contexts over many years to thousands of students. Then, we compared student explanations for each context to determine whether or not there were lines of reasoning common to contexts in both quantum and introductory physics.

Vectors

Is the magnitude of **A** + **B** *greater than, less than,* or *equal to* the magnitude **C** + **D**? (Each vector has the same length.)

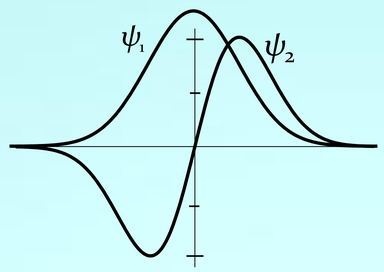


Answer	Common line(s) of reasoning	Example student explanation
Greater than (correct)	Most who answered correctly gave correct reasoning Some cited "vector addition" without further reasoning	*"The magnitude [of A + B] is larger since the vectors are more similar in direction."*
Less than	Students often appeared to apply an implicit or explicit method of vector *subtraction*	*"When you line the vectors up head to head and connect the ends with a line, [C + D's line] would be greater."*
Equal to	Students typically claimed that direction is not relevant and/or treated the vectors as *scalar* quantities	*"If the vectors are the same lengths then the magnitude of the resultant of each pair will be the same."*

We have asked many versions of this question: with physical context (*e.g.,* force, momentum, *etc.*) or without (as shown above), with or without grids, and some about addition of more than two vectors.

Student performance varied across course and/or context by as much as 20%. On some versions, as few as one-third of students answered correctly, even after 2-3 quarters of introductory physics instruction. However, the common lines of reasoning tended to be the same across all versions that we have asked.

Wave Functions



The two wave functions shown at left (ψ_1 and ψ_2) are entirely real at the instant shown. Consider a quantum particle described by the state $\psi_A = (\psi_1 + i\psi_2)/\sqrt{2}$ or by the state $\psi_B = (\psi_1 - \psi_2)/\sqrt{2}$. (Note: Each group of students was given either ψ_A or ψ_B, but not both.)

Is the probability that this particle is measured to be to the left of the origin *greater than, less than,* or *equal to* the probability that it is measured to be to the right of the origin?

Results for ψ_A

Answer	Common line of reasoning	Example student explanation
Equal to (correct)	Incomplete explanations based on the symmetry of the probability density for ψ_1 and ψ_2 were common	*"Because while ψ_1 is symmetric and ψ_2 is anti-symmetric, both $\lvert\psi_1\rvert^2$ and $\lvert\psi_2\rvert^2$ are symmetric."*
Greater than	Some students directly summed the two wave functions, ignoring the imaginary phase	*"If you add both ψ_1 and ψ_2, then square them, you will not get a symmetry about the y-axis."*

Results for ψ_B

Answer	Common line of reasoning	Example student explanation
Greater than (correct)	Most gave correct reasoning, often supported with a sketch	*"Subtract ψ_2 from ψ_1 and square it."*
Less than	Some students added the two wave functions, instead of subtracting	*"The amplitudes add on the right whereas on the left they oppose, making the amplitude smaller on the left."*
Equal to	Students often invoked symmetry; some explicitly identified that the negative sign in the definition of the wave function would have no impact on their answer	*"Equal, both [wave functions are] symmetric about the middle, so the positions of the particles will be symmetric about [the] middle. Negative doesn't matter, calculating probabilities squares away negative portions of [the] function."*

Versions of each question have been administered to students in introductory-, sophomore-, and junior-level courses that cover quantum mechanics. Some versions made use of wave functions that were not exactly the same as those given above, but that shared the same general shapes and symmetries. All versions were administered on exams after instruction.

As with the vectors questions, student performance varied across course and version. In some cases, as many as one-half to two-thirds of students gave explanations that were either incorrect or incomplete (*i.e.,* that lacked an important element of correct reasoning). Again, the common lines of reasoning were very similar from course to course and from version to version. Some incorrect lines of reasoning appeared to be related to underlying quantum concepts, not to vector superposition, and are not included in the tables above.

Patterns in Student Reasoning

Some of the incorrect lines of reasoning regarding quantum wave functions appear similar to the explanations given in response to questions about superposition of vectors.

- ~ It was common for students to believe that the relative direction between two vectors, or the relative phase between two wave functions, is irrelevant
- ~ Students often added instead of subtracting, and *vice versa*
- ~ Students who answered correctly often gave incomplete explanations (*e.g.,* referencing "vector addition" or "symmetry" without detailed reasoning)

Implications

Since the concepts necessary for adding wave functions and vectors are similar, what are the implications on further research and on instruction?

- → Students in quantum classes, who may have mastered addition of vectors, often make mistakes that are similar to those made in response to questions about vectors
- → Student understanding of wave function superposition might be improved by designing curriculum to help students reflect on the similarities between different physical contexts

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