Applying a symbolic forms lens to probability expressions in upper-division quantum mechanics

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What are the basic symbolic building blocks students use when generating and relating probability expressions in quantum mechanics?

How can the symbolic forms framework be adapted to help understand these building blocks?

Notations in quantum mechanics

- Dirac notation – linear algebra
  - State vector $|\psi\rangle$ describes general quantum state
  - Eigenvectors $|a_n\rangle$ of operator/observable $\hat{A}$ describe state after measurement of eigenvalue a_n
  - $|\langle a_n|\psi\rangle|^2$: probability of measuring eigenvalue a_n
- Wave function notation – differential equations
  - Wave function $\psi(x)$ describes general quantum state as probability amplitude distribution over space
  - Eigenfunctions $\varphi_{a_n}(x)$ of operators/observable $\hat{A}$ describe state after measurement of eigenvalue a_n
  - $\left|\int \varphi^*_{a_n}(x)\psi(x)dx\right|^2$: probability of measuring eigenvalue a_n
- Prior work in PER has explored general QM difficulties [1], affordances/limitations of [2] and preferences for [3] different QM notations, and proposals of symbolic forms [4] for some Dirac expressions in QM [5]

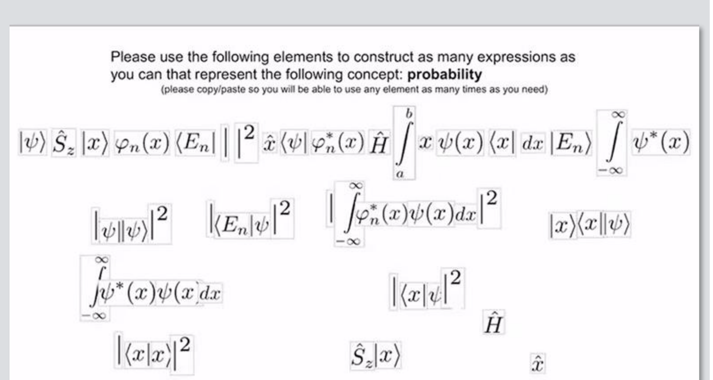
Symbolic forms [4]

- Proposed to discuss student reasoning about mathematical expressions
- Composed of a *symbol template* and a *conceptual schema*
  - Symbol template: the form a mathematical relationship takes
  - Conceptual schema: the meaning ascribed to that form
- Example: *parts-of-a-whole*
  - Symbol template: [□ + □ + □ ...]
  - Conceptual schema: individual elements that sum to create a larger whole

Methodology

Virtual individual interviews (N=2)

- Aliyah and Bilbo
- Task 1: Card-sorting task, multiple expressions
- Task 2: Expression construction task (shown right)



In-person pair interview (N_{Int}=1)

- Castor and Delilah
- Asked questions similar to homework/exams

Ex: "ISW potential: $|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$"

- "How would you find probability of it being in the left half of the well?"
- "How would you find probability of being in the lowest energy state?"

Dirac notation paired with vector ideas

Bilbo: (discussing sorted category containing $\vec{v}$, $\hat{j}$, $|\psi\rangle$, and $|E_n\rangle$): "to me, all vectors. Unit vector [$\hat{j}$], generic vector [$\vec{v}$], state- ... wave vector [$|\psi\rangle$], eigenstate vector [$|E_n\rangle$]."

Aliyah: (discussing $\langle\psi|\psi\rangle$) "why would I do ψ of ψ? Because physically like I'm thinking in terms of vectors it represents ψ along ψ. ... it's a traditional way to think about vectors like because our dot product represents – like $\vec{a}\cdot\vec{b}$ represents, basically, the projection of $\vec{a}$ along $\vec{b}$ or projection of $\vec{b}$ along $\vec{a}$."

Castor: (explaining why $\langle E_2|E_2\rangle = 1$): "Because, like 100% of E2 [gestures at the bra] is in the direction of E2 [gestures at the ket]."

"Vector in a space" and "projection" conceptual schemata, the former linked to $|\ \ \rangle$ and $\langle\ \ |$ symbol templates, and the latter to $\langle\ \ |\ \ \rangle$.

Dirac bras and kets paired with "quantum state"

Aliyah: "This [$|E_n\rangle$] represents a ket energy eigenstate, and this [$\langle E_n|$] represents a bra energy eigenstate. So these [$|\psi\rangle$ and $\langle\psi|$] are general ones, these [$|E_n\rangle$ and $\langle E_n|$] are specific energy eigenstates."

Bilbo: (discussing $|x\rangle$): "You could make x an eigenstate, you could make it a spin state ... put anything in there ... I just need it to be a ket."

Delilah: (discussing a superposition state written $|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$): "We just represent it as the probab– the square root of the probability times the first state [points to $|E_1\rangle$] plus the square root of probability times the second state [points to $|E_2\rangle$] plus the square root of the probability times the third state [points to $|E_3\rangle$]."

"Quantum state" conceptual schema paired with $|\ \ \rangle$ and $\langle\ \ |$ symbol templates.

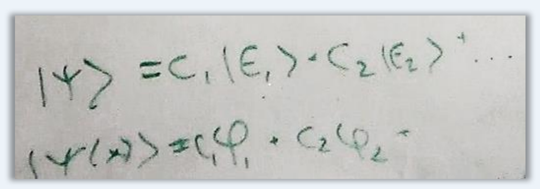
Function pairings with state ideas

Aliyah: "Those [$\psi(x)$, $\psi^*(x)$, $\varphi_n(x)$, and $\varphi^*_n(x)$] represent states ... some of them represent general states [$\psi(x)$ and $\psi^*(x)$], some of them represent specific energy states [$\varphi_n(x)$ and $\varphi^*_n(x)$], but they represent states."

Bilbo: "This [$\psi(x)$] is just another function, so what I'm thinking of is like an eigenstate [$\varphi_n(x)$] and just a generic state [$\psi(x)$]."

Delilah: "$\psi(x)$ ... is, like c_1 times $\varphi(x)$ [writes $\psi(x) = c_1\varphi_1(x) +$] ... I think these [points to $\varphi_1(x)$] are ... the energy eigenstates written in the position basis."

Castor: "These [φ_1 and φ_2 in $|\psi(x)\rangle = c_1\varphi_1 + c_2\varphi_2 +$] are the position eigenstates."



"Quantum state" conceptual schema with $f(a)$ and $f_n(a)$, also "function in a space" with $c_1f_1(a) + c_2f_2(a) + \cdots$ as a symbol template.

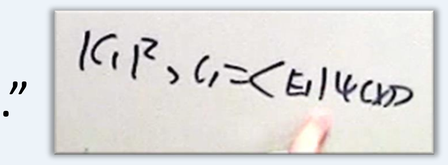
Other Dirac pairings

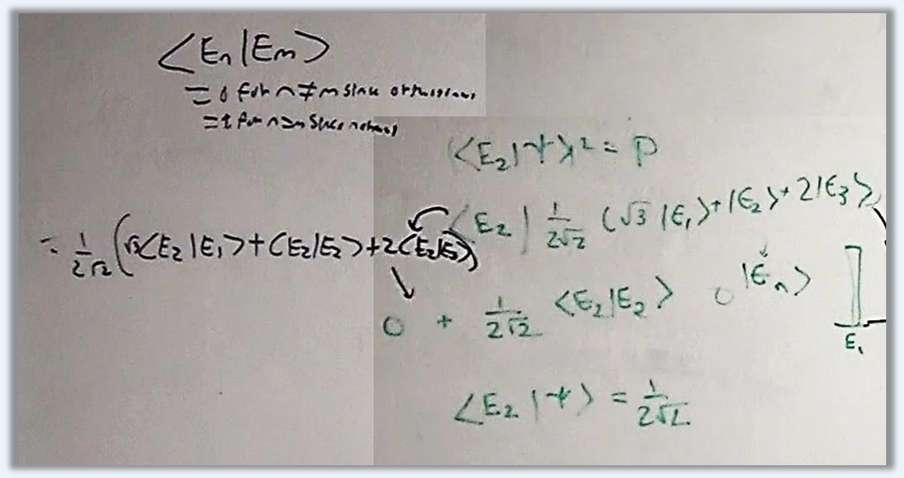
Aliyah: (explaining $\langle x|\psi\rangle$): "This will also represent the probability of finding x- sorry, the probability of finding the general state ψ in the eigenstate x"

Bilbo: (discussing $\langle\psi|\psi\rangle$): "That is just an inner product, though, I had been saying the inner product squared is a probability and that this [$\langle\psi|\psi\rangle$] is... just a density."

Castor: (explaining why a number they found was a probability): "Because it's the coefficient for the first energy state. ... we do the same thing as [writes $\langle E_1|\psi(x)\rangle$]."

Delilah: "Our probability for energy is the coefficient squared. And the coefficient is, E_1 times ψ [writes $|c_1|^2$, $c_1 = \langle E_1|\psi(x)\rangle$]."





"Probability"-like conceptual schema linked to $\langle\ \ |\ \ \rangle$, $|\langle\ \ |\ \ \rangle|^2$, and $|c_n|^2$ symbol templates.

Conceptual schema for "vector in a space" with $c_1|1\rangle + c_2|2\rangle + \cdots$ symbol template (*cf.* parts of a whole, basis expansion).

Other function pairings

Bilbo: "in this case [$\int \varphi^*_n(x)\psi(x)dx$] here ... you have this state [points to $\psi(x)$] ... and you want to ask the question of, you know, 'what about that state [$\psi(x)$] being in this [$\varphi^*_n(x)$] eigenstate.' ... I'm thinking, 'what is the projection of this eigenstate onto this wave function,' or maybe vice versa, but I don't think it should matter – dot products are ... commutative."

Delilah: [writes $\int_0^{\frac{L}{2}} \frac{1}{8}\left(\sqrt{3}\varphi_{E_1} + \varphi_{E_2} + 2\varphi_{E_3}\right)^2 dx$] "So we're – at every position we're computing [points to integrand] – like every infinitesimally small position we're computing the probability [again points to integrand]."

"Projection" conceptual schema tied to $\int f(a)g(a)da$ symbol template, and both "probability" and "probability density" tied to $\int |f(a)|^2da$ and $|f(a)|^2$, respectively.

Conceptual schemata	Associated symbol templates
Vector in a space	$|\ \ \rangle, \langle\ \ |, c_1|1\rangle + c_2|2\rangle + \cdots$
Function in a space	$c_1f_1(a) + c_2f_2(a) + \cdots$
Quantum state	$|\ \ \rangle, \langle\ \ |, f(a), f^*(a), f_n(a), f^*_n(a)$
Projection/dot product	$\langle\ \ |\ \ \rangle, \int f(a)g(a)da$
Probability density	$\langle\ \ |\ \ \rangle, |f(a)|^2$
Probability	$|c_n|^2, \langle\ \ |\ \ \rangle, |\langle\ \ |\ \ \rangle|^2,$ $\int f(a)g(a)da, \int |f(a)|^2da$

Conclusions

- Students developed a multitude of conceptual schemata and symbol templates
- Multiple overlapping pairs of templates schemata and templates were exhibited
  - Strict adherence to schema-template pairs as monolithic "symbolic forms" may not be most productive approach
- Previously-unreported symbol templates were found, including for Dirac inner products ($\langle\ \ |\ \ \rangle$) and generic (eigen-)functions ($f(a)$ and $f_n(a)$)

Future work

- Further analysis of the meaning students ascribe to QM expressions – identifying more symbol templates and their associated schemata
- Further investigation to answer the question: Do these shared conceptual schemata help explain the mechanism by which students are able to reason about and/or translate between notations in QM?

References

[1] C. Singh and E. Marshman, Review of student difficulties in upper-level quantum mechanics. *Phys. Rev. ST Phys. Educ. Res.* 11(2), 2015.

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